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Flags in zero dimensional complete intersection algebras and indices of real vector fields

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Abstract We introduce bilinear forms in a flag in a complete intersection local $\mathbb{R}$-algebra of dimension 0, related to the Eisenbud–Levine, Khimshiashvili bilinear form. We give a variational interpretation of these forms in terms of Jantzen's filtration and bilinear forms. We use the signatures of these forms to compute in the real case the constant relating the GSV-index with the signature function of vector fields tangent to an even dimensional hypersurface singularity, one being topologically defined and the other computable with finite dimensional commutative algebra methods.

Keywords Singularities of functions · Local algebra · Bilinear form · Index of vector field

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0 Introduction

Let $f_1, \dots, f_n : \mathbb{R}^n \to \mathbb{R}$ be germs of real analytic functions that form a regular sequence as holomorphic functions and let

$$\mathbf{A} := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_1, \dots, f_n)} \tag{1}$$

be the quotient finite dimensional algebra, where $\mathcal{A}_{\mathbb{R}^n,0}$ is the algebra of germs of real analytic functions on $\mathbb{R}^n$ with coordinates $x_1, \dots, x_n$. The class of the Jacobian

$$J = det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\dots,n}, \quad J_{\mathbf{A}} := [J]_{\mathbf{A}} \in \mathbf{A} \tag{2}$$

generates the socle (the unique minimal non-zero ideal) of the algebra $\mathbf{A}$. A symmetric bilinear form

$$\langle , \rangle_{L_{\mathbf{A}}} : \mathbf{A} \times \mathbf{A} \overset{\cdot}{\to} \mathbf{A} \overset{L_{\mathbf{A}}}{\to} \mathbb{R} \tag{3}$$

is defined by composing multiplication in $\mathbf{A}$ with any linear map $L_{\mathbf{A}} : \mathbf{A} \longrightarrow \mathbb{R}$ sending $J_{\mathbf{A}}$ to a positive number. The theory of Eisenbud–Levine and Khimshiashvili asserts that this bilinear form is nondegenerate and that its signature $\sigma_{\mathbf{A}}$ is independent of the choice of $L_{\mathbf{A}}$ (see [3,12]).

Let $f \in \mathbf{A}$ be an element in the maximal ideal. We define a flag of ideals in $\mathbf{A}$:

$$K_m := Ann_{\mathbf{A}}(f) \cap (f^{m-1}), \quad m \geq 1, \quad 0 \subset K_{\ell+1} \subset \cdots \subset K_1 \subset K_0 := \mathbf{A} \tag{4}$$

and a family of bilinear forms

$$\langle , \rangle_{L_{\mathbf{A}},f,m} : K_m \times K_m \to \mathbb{R}, \quad \langle a, a' \rangle_{L_{\mathbf{A}},f,m} = \left\langle \frac{a}{f^{m-1}}, a' \right\rangle_{L_{\mathbf{A}}}, \tag{5}$$

defined for $m = 0, \dots, \ell+1$. The division by f^{m-1} is defined up to elements in $Ann_{\mathbf{A}}(f^{m-1})$, but as $a' \in (f^{m-1})$, the last expression in (5) is well defined. We call the form $\langle , \rangle_{L_{\mathbf{A}},f,m}$, the order m bilinear form on the algebra $\mathbf{A}$, with respect to f. In Sect. 1 we prove:

Theorem 0.1 *For $m = 0, \dots, \ell + 1$ the order m bilinear form $\langle , \rangle_{L_{\mathbf{A}},f,m}$ on K_m induces a non-degenerate bilinear form*

$$\langle , \rangle_{L_{\mathbf{A}},f,m} : \frac{K_m}{K_{m+1}} \times \frac{K_m}{K_{m+1}} \longrightarrow \mathbb{R}, \tag{6}$$

whose signature $\sigma_{\mathbf{A},f,m}$ is independent of the linear map $L_{\mathbf{A}}$ chosen.

In Sect. 2 we give a variational interpretation of Theorem 0.1. Consider germs of analytic functions $f, f_1, f_2, \dots, f_n$ in $\mathbb{R}^n$ such that $f, f_2, \dots, f_n$ and $f_1, \dots, f_n$ are regular sequences as holomorphic functions. We consider the 1-parameter family of ideals $(f - t, f_2, \dots, f_n)$. Choose a small neighborhood $U_{\mathbb{C}}$ of $0 \in \mathbb{C}^n$ and a small $\varepsilon > 0$ such that:

(1) The sheaf of algebras on $U_{\mathbb{C}}$ defined by

$$\mathcal{B}_{\mathbb{C}} := \frac{\mathcal{O}_{U_{\mathbb{C}}}}{(f_2, \dots, f_n)}$$

is the structure sheaf of a 1-dimensional complete intersection $\mathbf{Z}_{\mathbb{C}} \subset U_{\mathbb{C}}$ such that the map

$$f : \mathbf{Z}_{\mathbb{C}} \to \Delta_{\varepsilon} \tag{7}$$

to the disk Δ_ε of radius ε in $\mathbb{C}$ is a finite analytic map, the sheaf $f_*\mathcal{B}_{\mathbb{C}}$ is a free $\mathcal{O}_{\Delta_\varepsilon}$-sheaf of rank ν and $f^{-1}(0) = 0$.

(2) $f_1|_{\mathbf{Z}_{\mathbb{C}}-\{0\}}$ is non-vanishing.

These conditions can be fulfilled due to the regular sequence hypothesis [4,10]. Denoting by $f_*\mathcal{B}^+$ the sheaf on $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ whose sections are the fixed points of the conjugation map $^- : f_*\mathcal{B}_{\mathbb{C}} \longrightarrow f_*\mathcal{B}_{\mathbb{C}}$, we have that $f_*\mathcal{B}^+$ is a free $\mathcal{A}_{(-\varepsilon,\varepsilon)}$-sheaf of rank ν. Its stalk over 0 is

$$\mathcal{B} := (f_*\mathcal{B})_0^+ = \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_2, \ldots, f_n)}.$$

Hence $\mathcal{B}$ is a free $\mathcal{A}_{\mathbb{R},0}$-module of rank ν. Introduce the 1-parameter family of $\mathbb{R}$-algebras obtained by evaluation

$$\mathbf{B}_{t_0} = f_*\mathcal{B}^+ \otimes_{\mathbb{R}} \frac{\mathbb{R}[t]_{(t-t_0)}}{(t-t_0)} = \left[\bigoplus_{p \in \mathbf{Z}_{\mathbb{C}} \cap f^{-1}(t_0)} \frac{\mathcal{O}_{\mathbb{C}^n,p}}{(f - t_0, f_2, \ldots, f_n)} \right]^+ . \tag{8}$$

$\mathbf{B}_0$ is a local algebra, $\mathbf{B}_{t_0}$ is a multilocal algebra and they form a vector bundle of rank ν over $(-\varepsilon, \varepsilon)$, whose sheaf of real analytic sections is $f_*\mathcal{B}^+$.

We define in the sheaf of sections $f_*\mathcal{B}^+$, a bilinear map

$$\langle, \rangle : f_*\mathcal{B}^+ \times f_*\mathcal{B}^+ \stackrel{\cdot}{\longrightarrow} f_*\mathcal{B}^+ \stackrel{\mathcal{L}}{\longrightarrow} \mathcal{A}_{(-\varepsilon,\varepsilon)}, \quad \langle a, b \rangle = \mathcal{L}(a \cdot b),$$

obtained by first applying the multiplication in the sheaf of algebras $f_*\mathcal{B}^+$ and then applying a chosen $\mathcal{A}_{(-\varepsilon,\varepsilon)}$-module map $\mathcal{L} : f_*\mathcal{B}^+ \longrightarrow \mathcal{A}_{(-\varepsilon,\varepsilon)}$ having the property that evaluating it at 0 gives a linear map $L_{\mathbf{B}_0} : \mathbf{B}_0 \longrightarrow \mathbb{R}$, verifying $L_{\mathbf{B}_0}([J]_{\mathbf{B}_0}) > 0$. The evaluation of $\langle, \rangle$ at a fiber $\mathbf{B}_t$ is a bilinear form defined on $\mathbf{B}_t$ and denoted by $\langle, \rangle_t$.

This family of non-degenerate bilinear forms is the usual tool in the Eisenbud–Levine and Khimshiashvili theory [3,12] to calculate the degree of the smooth map given by $(f, f_2, \ldots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$.

Define a sheaf map by multiplication with f_1

$$M_{f_1} : f_*\mathcal{B}^+ \longrightarrow f_*\mathcal{B}^+ \quad M_{f_1}(b) = f_1 b$$

and a family of bilinear maps, that we call relative:

$$\langle, \rangle^{rel} : f_*\mathcal{B}^+ \times f_*\mathcal{B}^+ \longrightarrow \mathcal{A}_{(-\varepsilon,\varepsilon)}, \quad \langle a, b \rangle^{rel} = \langle M_{f_1}(a), b \rangle \tag{9}$$

$$\langle, \rangle_t^{rel} : \mathbf{B}_t \times \mathbf{B}_t \longrightarrow \mathbb{R}, \quad \langle [a]_t, [b]_t \rangle_t^{rel} = \langle M_{[f_1]_t}([a]_t), [b]_t \rangle_t. \tag{10}$$

The bilinear forms $\langle, \rangle_t^{rel}$ are non-degenerate, for $t \neq 0$, having signature $\tau_\pm$, for $\pm t > 0$. The form $\langle, \rangle_t^{rel}$ degenerates for $t = 0$ on $Ann_{\mathbf{B}_0}([f_1]_{\mathbf{B}_0})$ [5]. Expanding in Taylor series at 0 the family of relative bilinear forms we arrive at the setting in [11,14], where it is shown how to obtain a flag of ideals

$$\ldots \subset \tilde{K}_r \subset \ldots \subset \tilde{K}_1 \subset \tilde{K}_0 = \mathbf{B}_0 \tag{11}$$

and bilinear forms in them and show how to reconstruct from the signatures τ_m of these bilinear forms the signatures $\tau_\pm$ (see Proposition 2.1). In our algebraic setting, the flag and the bilinear forms have the algebraic description:

Theorem 0.2 *For the family of bilinear forms $\langle, \rangle_t^{rel}$, in the family of algebras $\mathbf{B}_t$ (10) we have*

(1) *The set of* $b \in \mathcal{B}$ *such that the function* $t \to \langle [b]_t, [b']_t \rangle_t^{rel}$ *vanishes at* 0 *up to order* m*, for every* $b' \in \mathcal{B}$ *is the quotient ideal*

$$(f^m : f_1) := \{b \in \mathcal{B} \ / \ f_1 b \in (f^m)\} \subset \mathcal{B}$$

and

$$\tilde{K}_m = \frac{(f^m : f_1)}{(f) \cap (f^m : f_1)} \subset \frac{\mathcal{B}}{(f)} = \mathbf{B}_0. \tag{12}$$

(2) $(f) \cap (f^m : f_1) = M_f((f^{m-1} : f_1))$.

(3) *The bilinear form* $(b, b') \longrightarrow L_{\mathbf{B}_0}\left(\left[\frac{f_1 b}{f^m} b'\right]_{\mathbf{B}_0}\right)$

$$(f^m : f_1) \oplus (f^m : f_1) \xrightarrow{\frac{f_1}{f^m}\cdot} (f^m : f_1) \xrightarrow{\tilde{\pi}_0} \frac{\mathcal{B}}{(f)} = \mathbf{B}_0 \xrightarrow{L_{\mathbf{B}_0}} \mathbb{R}, \tag{13}$$

where $\tilde{\pi}_0$ *is the projection from* $\mathcal{B}$ *to* $\mathcal{B}/(f) = \mathbf{B}_0$*, vanishes on* $(f) \cap (f^m : f_1)$ *and induces Jantzen's bilinear form*

$$\langle , \rangle^m : \tilde{K}_m \otimes \tilde{K}_m \longrightarrow \mathbb{R} \qquad \langle , \rangle^m = \left\langle \frac{f_1 \cdot}{f^m}, \cdot \right\rangle_0, \tag{14}$$

giving the formula

$$\tau_+ = \sum_{m \geq 0} \tau_m, \quad \tau_- = \sum_{m \geq 0} (-1)^m \tau_m$$

In Sect. 3, we show

Theorem 0.3 *There is an isomorphism* $\varphi : \tilde{K}_1 \to K_1$*, induced by multiplication with the function* $\frac{f_1}{f}$*, which is sending the flag* $\{\tilde{K}_m\}_{m \geq 1}$ *in* $\mathbf{B}_0$ *in* (12) *to the flag* $\{K_m\}_{m \geq 1}$ *in* $\mathbf{A}$ *in* (4) *and Jantzen's bilinear forms* (14) *to the bilinear forms* (6)*. Hence, for* $m \geq 1$*, we have equal signatures* $\tau_m = \sigma_{\mathbf{A}, f, m}$ *and*

$$\tau_+ = \tau_0 + \sum_{m=1}^{\ell+1} \sigma_{\mathbf{A}, f, m}, \qquad \tau_- = \tau_0 + \sum_{m=1}^{\ell+1} (-1)^m \sigma_{\mathbf{A}, f, m}.$$

In Sect. 4 we apply these considerations for calculating indices of vector fields. If $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x_i}$ is a real analytic vector field with an algebraically isolated zero at 0 in $\mathbb{R}^n$, then the (Poincaré-Hopf) index of X at 0 is the signature of the bilinear form (3) constructed for the finite dimensional algebra

$$\mathbf{B} := \frac{\mathcal{A}_{\mathbb{R}^n,0}}{(X^1, \ldots, X^n)}, \qquad \langle , \rangle_{L_{\mathbf{B}}} : \mathbf{B} \times \mathbf{B} \xrightarrow{\cdot} \mathbf{B} \xrightarrow{L_{\mathbf{B}}} \mathbb{R}$$

where $L_{\mathbf{B}} : \mathbf{B} \to \mathbb{R}$ is a linear map with $L_{\mathbf{B}}(J_X) > 0$ (see [3,12]). Now assume further that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is a real analytic function, that X is tangent to the fiber $V_0 := f^{-1}(0)$, giving the relation $df(X) = hf$ with h a real analytic function called the cofactor. If 0 is a smooth point of V_0 then the signature $\sigma_{\mathbf{B}, h, 0}$ of the order 0 bilinear form

$$\langle , \rangle_{L,h,0} : \frac{\mathbf{B}}{Ann_{\mathbf{B}}(h)} \times \frac{\mathbf{B}_{\mathbb{R}}}{Ann_{\mathbf{B}}(h)} \xrightarrow{\cdot} \frac{\mathbf{B}}{Ann_{\mathbf{B}}(h)} \xrightarrow{L} \mathbb{R}$$

$$L : \frac{\mathbf{B}_{\mathbb{R}}}{Ann_{\mathbf{B}}(h)} \longrightarrow \mathbb{R}, \quad L\left(\frac{J_{\mathbf{B}}}{h}\right) > 0$$

is the Poincaré-Hopf index at 0 of the vector field $X|_{V_0}$, as can easily be deduced using the implicit function theorem. If 0 is an isolated critical point of V_0 and the dimension n of the ambient space is even, in [7] it is proved that

$$Ind_{V_{+,0}}(X) = Ind_{V_{-,0}}(X) = \sigma_{\mathbf{B},h,0} - \sigma_{\mathbf{A},h,0}. \tag{15}$$

If n is odd, it is proved in [6] that

$$Ind_{V_{\pm,0}}(X) = \sigma_{\mathbf{B},h,0} + K_{\pm}. \tag{16}$$

In the case of odd dimensional ambient space and for f a germ of a real analytic function with an algebraically isolated singularity at 0, we calculate the constants $K_{\pm}$ by studying the family of contact vector fields

$$X_t = (f-t)\frac{\partial}{\partial x_1} + \sum_{i=1}^{N}\left[\frac{\partial f}{\partial x_{2i+1}}\frac{\partial}{\partial x_{2i}} - \frac{\partial f}{\partial x_{2i}}\frac{\partial}{\partial x_{2i+1}}\right],$$

where $f_j := \frac{\partial f}{\partial x_j}$. For $t \neq 0$, the signatures of the relative bilinear forms correspond to the sum of the Poincaré-Hopf indices of the restriction of X_t to V_t. Our transport from the algebra $\mathbf{B}_0$ to the Jacobian algebra $\mathbf{A}$ is a local analogue of the Poincaré-Hopf Theorem relating information of the singular point of X to invariants of the singularity of f.

Using these explicit computations for contact vector fields, we conclude the search for an algebraic formula for the real GSV-index using local algebra by determining the values of the constants $K_{\pm}$:

Theorem 0.4 *Let V be an algebraically isolated hypersurface singularity in $\mathbb{R}^{2N+1}$, then the constants $K_{\pm}$ in (16) relating the GSV-index and the signature $\sigma_{\mathbf{B},h,0}$ are*:

$$K_+ = \sum_{m\geq 1}\sigma_{\mathbf{A},f,m}, \quad K_- = \sum_{m\geq 1}(-1)^m\sigma_{\mathbf{A},f,m}.$$

1 Higher order signatures in A

We use the definitions and notations of Sect. 0. The algebra $\mathbf{A}$ has an intrinsic $\mathbf{A}$-valued bilinear map, which is the multiplication in $\mathbf{A}$:

$$(\ ,\ )_{\mathbf{A}} : \mathbf{A}\times\mathbf{A} \longrightarrow \mathbf{A} \qquad (a,b)_{\mathbf{A}} := ab. \tag{17}$$

In terms of this pairing and the non-singular pairing $\langle,\rangle_{L_{\mathbf{A}}}$ in (3), the orthogonal of an ideal $I \subset \mathbf{A}$ is the annihilator ideal in the algebra $\mathbf{A}$: $I^{\perp} = Ann_{\mathbf{A}}(I)$. The process of taking the orthogonal induces an involution in the set of ideals of $\mathbf{A}$, which is reversing the natural inclusions of sets in $\mathbf{A}$. In particular, the orthogonal to the maximal ideal is the socle. It is 1-dimensional and the class of the Jacobian $J_{\mathbf{A}}$ is a generator (see [3,4,12]).

Choose now an element $f \in \mathbf{A}$ in the maximal ideal. Consider the linear map induced in $\mathbf{A}$ by multiplication with f:

$$M_f : \mathbf{A} \longrightarrow \mathbf{A} \qquad M_f(a) = fa.$$

For $j \geq 1$, the maps M_f^j are selfadjoint maps for the bilinear map (17):

$$(M_f^j a, b)_{\mathbf{A}} = f^j ab = af^j b = (a, M_f^j b)_{\mathbf{A}},$$

and hence they are also selfadjoint maps for the bilinear form $\langle , \rangle_{L_{\mathbf{A}}}$. We have that

$$Ann_{\mathbf{A}}(f^j) = Ker(M_f^j) \quad \text{and} \quad (f^j) = Im(M_f^j)$$

and each of these spaces is the orthogonal of the other in $\mathbf{A}$, since M_f^j is selfadjoint.

Consider the flag of ideals in $\mathbf{A}$

$$0 \subset (f^\ell) \subset (f^{\ell-1}) \subset \cdots \subset (f^2) \subset (f) \subset \mathbf{A}, \tag{18}$$

where ℓ is minimal with $f^{\ell+1} = 0$ and the orthogonal flag of ideals

$$0 \subset Ann_{\mathbf{A}}(f) \subset Ann_{\mathbf{A}}(f^2) \subset \cdots \subset Ann_{\mathbf{A}}(f^{\ell-1}) \subset Ann_{\mathbf{A}}(f^\ell) \subset \mathbf{A}. \tag{19}$$

The linear map $M_f : \mathbf{A} \longrightarrow \mathbf{A}$ is a nilpotent map $M_f^{\ell+1} = 0$.

Lemma 1.1 *For $j = 1, \ldots, \ell + 1$, there are linear subspaces P_j of $\mathbf{A}$, called primitive subspaces, such that*

$$\mathbf{A} = \bigoplus_{j=1}^{\ell+1} \left[\bigoplus_{k=0}^{j-1} M_f^k P_j \right], \tag{20}$$

with $M_f^{j-1} : P_j \longrightarrow \mathbf{A}$ injective and $M_f^j(P_j) = 0$. The mapping $M_f : \mathbf{A} \to \mathbf{A}$ is in Jordan canonical form in any basis obtained by choosing bases of each of the spaces P_j and extending them to a basis of $\mathbf{A}$ by the action of M_f as in (20).

Proof We recall how to choose a basis of $\mathbf{A}$ as a vector space over $\mathbb{R}$ that expresses M_f in Jordan canonical form. Inductively, let us begin by choosing linearly independent vectors $v_1, \ldots, v_{n_{\ell+1}}$ generating a vector space $P_{\ell+1}$ complementary to $Ann_{\mathbf{A}}(f^\ell)$ in $\mathbf{A}$ and choose as first vectors of a basis of $\mathbf{A}$ the vectors

$$\{v_j, f v_j, \ldots, f^\ell v_j\}_{j=1,\ldots,n_{\ell+1}}.$$

With $P_{\ell+1}$ we construct the Jordan blocks of maximal size ℓ of M_f. Then, we choose linearly independent vectors $v_{n_{\ell+1}+1}, \ldots, v_{n_{\ell+1}+n_\ell}$ generating a vector space P_ℓ with the property that

$$Ann_{\mathbf{A}}(f^{\ell-1}) \oplus M_f(P_{\ell+1}) \oplus P_\ell = Ann_{\mathbf{A}}(f^\ell).$$

We choose the next part of the basis by choosing the vectors

$$\{v_j, f v_j, \ldots, f^{\ell-1} v_j\}_{j=n_{\ell+1}+1,\ldots,n_{\ell+1}+n_\ell}$$

to construct the Jordan blocks of size $\ell - 1$, and so on. The space of 1-st primitive vectors P_1 is formed of vectors in $\mathbf{A}$ with the property that

$$M_f^\ell(P_{\ell+1}) \oplus M_f^{\ell-1}(P_\ell) \oplus \cdots \oplus M_f^2(P_3) \oplus M_f(P_2) \oplus P_1 = Ann_{\mathbf{A}}(f).$$

□

We call the vectors in P_j jth-primitive vectors, and we denote by n_j the dimension of P_j. Hence n_j is also the number of Jordan blocks of size j in M_f. It is convenient to present the direct sum decomposition (20) by the matrix:

$$\mathbf{A}=\begin{pmatrix} P_1 & P_2 & P_3 & P_4 & \cdots & P_\ell & P_{\ell+1}\\ 0 & M_fP_2 & M_fP_3 & M_fP_4 & \cdots & M_fP_\ell & M_fP_{\ell+1}\\ 0 & 0 & M_f^2P_3 & M_f^2P_4 & \cdots & M_f^2P_\ell & M_f^2P_{\ell+1}\\ & & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & M_f^{\ell-1}P_\ell & M_f^{\ell-1}P_{\ell+1}\\ 0 & 0 & 0 & 0 & \cdots & 0 & M_f^{\ell}P_{\ell+1}\end{pmatrix}, \tag{21}$$

meaning that an element of $\mathbf{A}$ has components in the form of an upper triangular matrix where the (i, j)th-entry of the matrix is an arbitrary element in $M_f^{i-1}(P_j)$, with $i, j = 1, \dots, \ell+1$. Each column is formed by equidimensional subspaces, until we reach the zero subspace, and the map M_f acts as a map preserving columns and descending one row. Hence, restricting to a column in (21), the map M_f is an isomorphism until it reaches the diagonal, where M_f is the zero map.

Using this representation for $\mathbf{A}$ and recalling the flag of ideals (4), we have

Lemma 1.2 (1) *The ideal (f^m) is formed by the last $\ell+1-m$ rows of the matrix* (21).

(2) *Its orthogonal $Ann_{\mathbf{A}}(f^m)$ is formed by the elements in a band of width m above the diagonal in* (21)*, including the diagonal.*

(3) *The ideal K_m in* (4) *is formed by the lower $\ell+2-m$ diagonal terms.*

(4) *The ideal $K_m^\perp$, orthogonal to K_m is*

$$K_m^\perp = (f) + Ann_{\mathbf{A}}(f^{m-1}). \tag{22}$$

Example 1.1 For $\ell = 3$ and $m = 3$, we have

$$(f^2)=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & M_f^2P_3 & M_f^2P_4\\ 0 & 0 & 0 & M_f^3P_4\end{pmatrix}\quad Ann_{\mathbf{A}}(f^2)=\begin{pmatrix} P_1 & P_2 & 0 & 0\\ 0 & M_fP_2 & M_fP_3 & 0\\ 0 & 0 & M_f^2P_3 & M_f^2P_4\\ 0 & 0 & 0 & M_f^3P_4\end{pmatrix}$$

$$K_3=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & M_f^2P_3 & 0\\ 0 & 0 & 0 & M_f^3P_4\end{pmatrix};\ (f)+Ann_{\mathbf{A}}(f^2)=\begin{pmatrix} P_1 & P_2 & 0 & 0\\ 0 & M_fP_2 & M_fP_3 & M_fP_4\\ 0 & 0 & M_f^2P_3 & M_f^2P_4\\ 0 & 0 & 0 & M_f^3P_4\end{pmatrix}=K_3^\perp.$$

Proof of Lemma 1.2 Since M_f corresponds to going down 1 row in (21), parts 1, 2 and 3, are clear. To prove part 4, note first that

$$(f) + Ann_{\mathbf{A}}(f^{m-1}) \subset K_m^\perp.$$

The ideal (f) is given by all the terms in (21), except for the first row. Since $Ann_{\mathbf{A}}(f^{m-1})$ is the band matrix above the diagonal of width $m-1$, we obtain that the only contribution of $(f)+Ann_{\mathbf{A}}(f^{m-1})$ to (f) is given by the first $m-1$ terms in the first row. On the other hand $K_m = Ann_{\mathbf{A}}(f) \cap (f^{m-1})$ consist of the last $\ell+2-m$ terms in the diagonal. We observe on using (21) that the ideals $(f) + Ann_{\mathbf{A}}(f^{m-1})$ and K_m have complementary dimensions in $\mathbf{A}$. Now (22) must hold, as the bilinear form $\langle , \rangle_{L_{\mathbf{A}}}$ is non-degenerate. □

Proposition 1.1 *For the bilinear forms in* (5)*, we have*

(1) $\langle , \rangle_{L_{\mathbf{A}},f,0} = \langle f\cdot, \cdot\rangle_{L_{\mathbf{A}}}$ *has* $K_1 = Ann_{\mathbf{A}}(f)$ *as degeneracy locus and the induced non-degenerate bilinear form in* $\frac{\mathbf{A}}{Ann_{\mathbf{A}}(f)}$ *is obtained by choosing* $\frac{J_{\mathbf{A}}}{f}$ *as generator of the 1 dimensional socle of* $\frac{\mathbf{A}}{Ann_{\mathbf{A}}(f)}$ *and defining the bilinear form as multiplication followed by a real valued map sending* $\frac{J_{\mathbf{A}}}{f}$ *to a positive number.*

(2) *The bilinear form* $\langle , \rangle_{L_{\mathbf{A}},f,1} = \langle , \rangle_{L_{\mathbf{A}}}|_{K_1\times K_1}$ *has* K_2 *as degeneracy locus.*

(3) *For* $m \geq 2$ *the bilinear form* $\langle , \rangle_{L_{\mathbf{A}},f,m}$ *in* (5) *is well defined and has* K_{m+1} *as degeneracy locus.*

Proof (1) The inner product $\langle f\cdot, \cdot\rangle_{L_{\mathbf{A}}}$ vanishes on $Ann_{\mathbf{A}}(f)$. If $\langle fa, a'\rangle_{L_A} = 0$ for all a', then $fa = 0$, since (3) is a non-degenerate bilinear form on $\mathbf{A}$ [3,4,12]. Hence, $\langle , \rangle_{L_{\mathbf{A}},f,0}$ has K_1 as degeneracy locus. The algebra $\frac{\mathbf{A}}{Ann_{\mathbf{A}}(f)}$ has a one-dimensional socle generated by the class of $J_{\mathbf{A}}/f$ (see [5] for more details).

(2) Note first that $K_2 = Ann_{\mathbf{A}}(f)\cap(f) = K_1\cap K_1^{\perp}$, by (4) and (22). Hence, given $a \in K_2$ and any $b \in K_1$, it follows that $(a, b)_{\mathbf{A}} = 0$, so K_2 is contained in the degeneracy locus of $\langle , \rangle_{L,f,1}$. On the other hand, let $a \in K_1 - K_2 = K_1 - K_1^{\perp}$. Then aK_1 is a non-zero ideal in $\mathbf{A}$, and so contains the socle of $\mathbf{A}$. We obtain an expression $J_{\mathbf{A}} = ac$, for some $c \in K_1$. Hence, $\langle a, c\rangle_{L_{\mathbf{A}}} = L_{\mathbf{A}}(ac) = L_{\mathbf{A}}(J_{\mathbf{A}}) > 0$, so that a is not in the degeneracy locus of $\langle , \rangle_{L,f,1}$.

(3) Let $m \geq 2$. We first show that the bilinear form $\langle , \rangle_{L_{\mathbf{A}},f,m}$ is well defined, i.e. is independent of the division by f^{m-1} in K_m. Let a, b be in $K_m = Ann_{\mathbf{A}}(f)\cap(f^{m-1})$. Then there exists $a_1 \in \mathbf{A}$ such that $a = a_1 f^{m-1}$ and $\langle a, b\rangle_{L_{\mathbf{A}},f,m} = \langle a_1, b\rangle_{L_{\mathbf{A}}}$. Let also $a = a_2 f^{m-1}$. Then $\langle a_1, b\rangle_{L_{\mathbf{A}}} = \langle a_2, b\rangle_{L_{\mathbf{A}}}$, because $a_1 - a_2 \in Ann_{\mathbf{A}}(f^{m-1})$ and $b \in (f^{m-1})$.

If $a \in K_{m+1}$, then $\frac{a}{f^{m-1}} \in (f)$, and since $b \in K_m \subset Ann_{\mathbf{A}}(f)$, we have $\frac{a}{f^{m-1}}b = 0$. Hence, the form $\langle , \rangle_{L_{\mathbf{A}},f,m}$ degenerates on K_{m+1}.

Let $a \in K_m - K_{m+1}$. In order to prove that the form $\langle , \rangle_{L_{\mathbf{A}},f,m}$ is non-degenerate on a, we have to show that $\frac{a}{f^{m-1}} \notin K_m^{\perp}$. Using the representation (21), and part (3) of Lemma 1.2, the $a_{m,m}$ entry in a is not zero, and $a_{m,m} \in M_f^{m-1}P_m$. Now $\frac{a}{f^{m-1}}$ is obtained by lifting all the elements in the representation by $m-1$ rows, keeping the columns fixed. We observe that $\frac{a}{f^{m-1}} \notin (f)$. It now suffices to show that $\frac{a}{f^{m-1}} \notin Ann_{\mathbf{A}}(f^{m-1})$. But by part (4) of Lemma 1.2, the space $Ann_{\mathbf{A}}(f^{m-1})$ is given by the band matrix of width $m-1$, including the diagonal. Hence, $\frac{a}{f^{m-1}}$ is not an element of $Ann_{\mathbf{A}}(f^{m-1})$. □

Proof of Theorem 0.1 By Proposition 1.1, we have that K_{m+1} is the locus of the bilinear form $\langle , \rangle_{L_{\mathbf{A}},f,m}$, so that K_m/K_{m+1} inherits a non-degenerate bilinear form. The linear forms $L_{\mathbf{A}}$, verifying $L_{\mathbf{A}}(J_{\mathbf{A}}) > 0$ form an open connected set in the dual space $\mathbb{R}^{n^*}$. The signature is an integer valued continuous function of $L_{\mathbf{A}}$, hence it is constant. □

Corollary 1 *For* $m \geq 1$*, the mapping*

$$M_f^{m-1} : P_m \longrightarrow K_m/K_{m+1} \tag{23}$$

is a well defined isomorphism. The pairing of m-primitive vectors

$$\langle , \rangle_{L_{\mathbf{A}},m}^{prim} : P_m \times P_m \longrightarrow \mathbb{R}, \qquad \langle a, b\rangle_{L_{\mathbf{A}},m}^{prim} := \langle M_f^{m-1}a, b\rangle_{L_{\mathbf{A}}}$$

is a non-degenerate symmetric bilinear pairing, induced by the pairing (6) *via the isomorphism* (23)*.*

Proof Using the representation (21) for the elements of $\mathbf{A}$ and the description of K_m given in part 3 of Lemma 1.2, we have that $M_f^{m-1} : P_m \longrightarrow K_m$ is injective, and hence (23) is a well defined isomorphism. The pull back of the non-degenerate bilinear form on K_m/K_{m+1} via this last map is

$$\begin{aligned}\langle b, b'\rangle_{L_{\mathbf{A}},m} &= L_{\mathbf{A}}(f^{m-1}\cdot b\cdot b') = L_{\mathbf{A}}\left(\left(\frac{1}{f^{m-1}}\cdot f^{m-1}b\right)\cdot f^{m-1}b'\right)\\ &= \langle M_f^{m-1}(b), M_f^{m-1}(b')\rangle_{L_{\mathbf{A}},f,m}.\end{aligned}$$

□

Example 1.2 Let f be a germ of a real analytic function in $\mathbb{R}^n$, with an algebraically isolated critical point. This means that the ideal generated by the partial derivatives of f in the ring of germs of holomorphic functions has finite codimension. Let $\mathbf{A} = \mathbf{A}(f)$ be given by (1), with $f_i := \frac{\partial f}{\partial x_i}$ and let ℓ be as in (18). Let $\sigma_{f,m} = \sigma_{\mathbf{A}(f),f,m}$, $m = 0, \ldots, \ell+1$, be the signatures given by Theorem 0.1. These are invariants associated to the germ f. We call $\sigma_{f,m}$ the order m signature of f.

2 The family of bilinear forms in $\mathcal{B}$

In this section we construct a family of bilinear forms $\langle,\rangle^{rel}$, that we call relative, which is constructed from the equations

$$f - t = f_2 = \cdots = f_n = 0$$

which are non-degenerate for $t \neq 0$. We do Taylor series expansion of $\langle,\rangle^{rel}$ and determine an algebraic procedure to compute the signatures for $t \neq 0$ in terms of local linear algebra in the ring $\frac{\mathcal{A}_{\mathbb{R}^n,0}}{(f_2,\ldots,f_n)}$ via the first terms of the above Taylor series expansion.

Recall the setting and definitions of Sect. 0. Note in particular that since the map in (7) is a finite analytic map, the inverse image $f^{-1}((-\varepsilon,\varepsilon))$ is a finite union of curves (parameterized by $(-\varepsilon, 0]$ or $[0,\varepsilon)$), which come together at 0. The conjugation map permutes them, and the fixed components correspond to $\mathbf{Z} := \mathbf{Z}_{\mathbb{C}} \cap \mathbb{R}^n$. Hence $\mathbf{Z}$ consists either of 0 only or of a finite number of these real curves all passing through 0, which is its only singular point. Note that the degree of the covering map $f : \mathbf{Z} - \{0\} \longrightarrow (-\varepsilon,\varepsilon) - \{0\}$ may be distinct for $t > 0$ and $t < 0$. In the sheaf $f_*\mathcal{B}^+$ we have information about the points $\{f = t\}_{t\in(-\varepsilon,\varepsilon)} \cap \mathbf{Z}_{\mathbb{C}}$ in $U_{\mathbb{C}}$, real or complex.

Lemma 2.1 *The signature of the non-degenerate bilinear forms* $\langle,\rangle_t$ *on* $\mathbf{B}_t$ *is independent of t and it is equal to the sum of the signatures of the bilinear forms computed on the local rings* $\mathbf{B}_{t,p}$ *for* $p \in \mathbf{Z} \cap f^{-1}(t)$*, for each* $t \in (-\varepsilon,\varepsilon)$.

Proof This is the usual procedure due to Eisenbud–Levine and Khimshiashvili [3,12] to calculate the degree applied to the smooth map given by $(f, f_2, \ldots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$. In particular, the contribution to the signature coming from points in $\mathbf{Z}_{\mathbb{C}} - \mathbf{Z}$ is always 0. □

Using a trivialization of $f_*\mathcal{B}^+$, we can transfer the relative forms $\langle,\rangle_t^{rel}$ from $\mathbf{B}_t$ to $\mathbf{B}_0$. So we have a family of bilinear forms that we denote by $\langle,\rangle_t$. We are interested in the Taylor expansion of this family of bilinear forms at $t = 0$. We will use the following Proposition, containing results from [11,14]:

Proposition 2.1 *Let* $\langle,\rangle_t$, $t \in (-\varepsilon,\varepsilon)$*, be an analytic family of forms on a finite dimensional vector space* $\mathbf{B}_0$*. Assume that the forms* $\langle,\rangle_t$ *are nondegenerate for* $t \neq 0$*. Let* $\tilde{K}_i$, $i = 0, \ldots, r$, *be the set of* $[b]_0 \in \mathbf{B}_0$*, such that the functions* $t \mapsto \langle [b]_0, [b']_0\rangle_t$ *vanish at* 0 *up to order* i *for any* $b' \in \mathcal{B}$*. Then*

(1) *For* $i = 0, \dots, r$*, a bilinear form* $\langle , \rangle^i$*, is well defined on* $\tilde{K}_i$ *by*

$$\langle [b]_0, [b']_0\rangle^i = \frac{1}{i!}\frac{d^i}{dt^i}\langle b, b'\rangle_t|_{t=0}. \tag{24}$$

(2) *The bilinear form* $\langle , \rangle^i$ *degenerates on* $\tilde{K}_{i+1}$*, and induces a nondegenerate bilinear form on* $\tilde{K}_i/\tilde{K}_{i+1}$*. Denote its signature by* τ_i*.*

(3) *The signatures* τ_+ *and* τ_- *of the forms* $\langle , \rangle_t$ *on* $\mathbf{B}_0$*, for* $t > 0$ *and* $t < 0$*, respectively, are given by*

$$\tau_+ = \sum_{i=0}^{r} \tau_i, \qquad \tau_- = \sum_{i=0}^{r} (-1)^i \tau_i. \tag{25}$$

Proof of Theorem 0.2 (1) Let $b \in \mathcal{B} - \{0\}$. Then, there is a unique integer j and $c \in \mathcal{B}$ with $[c]_{\mathbf{B}_0} \neq 0$ such that $f_1 b = f^j c$. Since $[c]_{\mathbf{B}_0} \neq 0$, and $[J]_{\mathbf{B}_0}$ is a generator of the socle of $\mathbf{B}_0$ we may find $e_0 \in \mathbf{B}_0$ such that $[c]_{\mathbf{B}_0} e_0 = [J]_{\mathbf{B}_0}$. Choose any $e \in \mathcal{B}$ with the property that $[e]_{\mathbf{B}_0} = e_0$, so that $\mathcal{L}(ce)(0) = L_{\mathbf{B}_0}([c]_{\mathbf{B}_0} e_0) = L_{\mathbf{B}_0}([J]_{\mathbf{B}_0}) \neq 0$. For any $b' \in \mathcal{B}$, we have $f_1 b b' = f^j c b' \in (f^j)$. Hence,

$$\langle b, b'\rangle^{rel} = \mathcal{L}(f_1 b b') = \mathcal{L}(f^j c b') = t^j \mathcal{L}(cb') \in (t^j)$$

and

$$\langle b, e\rangle^{rel} = \mathcal{L}(f_1 b e) = \mathcal{L}(f^j c e) = t^j \mathcal{L}(ce) \in (t^j) - (t^{j+1}).$$

Hence, if b is as in the statement of part (1), we have that $j \geq m$ and hence $f_1 b \in (f^m)$, i.e. $b \in (f^m : f_1)$. This proves the first assertion. The second assertion follows from the first by evaluating it at $t = 0$ and using (8).

(2) Let $b \in (f) \cap (f^m : f_1)$. Then $b = cf$ and $(cf) f_1 = e f^m$, so that $c f_1 = e f^{m-1}$. Hence, $c \in (f^{m-1} : f_1)$ and $b = cf \in M_f(f^{m-1} : f_1)$. The converse is obvious.

(3) Let $b \in (f) \cap (f^m : f_1)$ and $b' \in (f^m : f_1)$. Then

$$\left(\frac{f_1 b}{f^m}\right) b' = b\left(\frac{f_1 b'}{f^m}\right) \in (f),$$

since $b \in (f)$. Hence $[\langle \frac{f_1 b}{f^m}, b'\rangle]_{\mathbf{B}_0} = 0$ and the bilinear form in (13) vanishes on $(f) \cap (f^m : f_1)$. Taking the quotient by $(f) \cap (f^m : f_1)$, we obtain by part (1) that it is a bilinear form defined on $\tilde{K}_m$ and it has the same expression as Jantzen's form, since $f^m = t^m$, so they coincide. □

3 Transporting the signatures to the algebra A

The aim of this section is to establish a relationship between the higher order bilinear forms $\langle , \rangle_{L_A, f, m}$ (5) and their signatures $\sigma_{\mathbf{A}, f, m}$ in the algebra $\mathbf{A}$ and Jantzen's relative forms $\langle , \rangle^m$ (24) and their signatures τ_m in $\mathbf{B}_0$.

Define the isomorphism of $\mathcal{B}$-modules

$$\begin{array}{ccc} \mathcal{B} & & \mathcal{B} \\ \cup & & \cup \\ (f : f_1) & \xrightarrow{\Phi} & (f_1 : f) \end{array}$$

$$\Phi(b) = \frac{bf_1}{f}, \quad \Phi^{-1}(c) = \frac{cf}{f_1}$$

Lemma 3.1 *The isomorphism Φ induces isomorphisms of $\mathcal{B}$-modules, for $m \geq 1$:*

$$\begin{gathered} \Phi : (f^m : f_1) \longrightarrow (f_1 : f) \cap (f^{m-1}) \\ \Phi : (f^m) \longrightarrow (f^{m-1} f_1) \\ \varphi : \tilde{K}_1 = Ann_{\mathbf{B}_0}(f_1) \longrightarrow K_1 = Ann_{\mathbf{A}}(f). \end{gathered} \tag{26}$$

Proof If $b \in (f^m : f_1)$, then there exists $c \in \mathcal{B}$ such that $bf_1 = cf^m$. Hence

$$\Phi(b) = \frac{bf_1}{f} = cf^{m-1} \in (f_1 : f) \cap (f^{m-1}).$$

Conversely, if $c = df^{m-1} \in (f_1 : f)$, then

$$df^m = cf = ef_1 \quad \Rightarrow \quad e = \Phi^{-1}(c) \in (f^m : f_1).$$

This proves the first assertion. The second one is just $\Phi(bf^m) = bf^{m-1} f_1$. The third assertion is obtained by taking the quotient of the first assertion in the Lemma by the second relation in the case $m = 1$. □

Let $f_2, \ldots, f_n$ be a regular sequence of holomorphic functions, denote the volume form by $dVol = dx_1 \wedge \cdots \wedge dx_n$, and let $\mathbf{Z}_{\mathbb{C}}$ be the complete intersection $f_2 = \ldots = f_n$ as in Sect. 2. For any holomorphic function g define the Jacobian of g by

$$dg \wedge df_2 \wedge \ldots \wedge df_n := Jac(g)\, dVol, \quad Jac(g) = \begin{vmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdot & \cdots & \cdot \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

Recall the construction of the generator of the Rosenlicht differentials (see [13]) or dualizing module, which is a rational differential form ω_0 on $\mathbf{Z}_{\mathbb{C}}$ having the property

$$\omega_0 \wedge (df_2 \wedge \ldots \wedge df_n)|_{\mathbf{Z}_{\mathbb{C}}} = dVol|_{\mathbf{Z}_{\mathbb{C}}} \in \frac{\Omega^n_{\mathbb{C}^n}}{(f_2, \ldots, f_n)\Omega^n_{\mathbb{C}^n}}.$$

The dualizing module on $\mathbf{Z}_{\mathbb{C}}$ is then $\mathcal{O}_{\mathbf{Z}_{\mathbb{C}}}\omega_0$, and it consists of all rational differential forms σ on $\mathbf{Z}_{\mathbb{C}}$ that have the property that the residue at 0 of $h\sigma$ is 0, for any holomorphic function h on $\mathbf{Z}_{\mathbb{C}}$. Recall also that the residue of a differential form σ at $0 \in \mathbf{Z}_{\mathbb{C}}$ is the sum of the residues of the rational differential form $\nu^*\sigma$ at $\nu^{-1}(0)$, where ν is the normalization map of $\mathbf{Z}_{\mathbb{C}}$. Directly from the definitions above, one obtains that for any holomorphic function g on $\mathbb{C}^n$

$$d(g|_{\mathbf{Z}_{\mathbb{C}}}) = Jac(g)|_{\mathbf{Z}_{\mathbb{C}}}\omega_0.$$

Note that the logarithmic derivative of $g|_{\mathbf{Z}_{\mathbb{C}}}$ is $\frac{Jac(g)}{g}\omega_0$ and its residue at 0 is the sum of the vanishing orders of the function $g \circ \nu$ at $\nu^{-1}(0)$, and hence a positive integer.

Lemma 3.2 *Let $f, f_1, \ldots, f_n$, and φ be as in Sect. 2. Let $J_{\mathbf{B}_0}$ and $J_{\mathbf{A}}$ be the Jacobians of $(f, f_2, \ldots, f_n)$ and $(f_1, \ldots, f_n)$ respectively. Then there exists a positive constant $c = c(f)$ such that $\varphi(J_{\mathbf{B}_0}) = cJ_{\mathbf{A}}$.*

Proof Since $([f]_{\mathbf{A}}) \subsetneq \mathbf{A}$, then taking orthogonal of this relation, we obtain that the ideal K_1 is not the 0-ideal, and hence K_1 contains the socle. Since $\varphi : \tilde{K}_1 \longrightarrow K_1$ is an isomorphism of non-zero ideals, each containing its corresponding 1-dimensional socle, the map φ sends the socle ideal to the corresponding socle ideal. Hence φ sends the Jacobian of $\mathbf{B}_0$ to a non-zero multiple of the corresponding Jacobian of $\mathbf{A}$.

Thus we know that there is a non-zero real number c with the property

$$\left[\frac{f_1 Jac(f)}{f}\right]_{\mathbf{A}} = c[Jac(f_1)]_{\mathbf{A}}.$$

Hence there is a holomorphic function h on $\mathbf{Z}_{\mathbb{C}}$ with the property

$$\frac{f_1 Jac(f)}{f} - cJac(f_1) = hf_1$$

Dividing by f_1 and multiplying by ω_0 we obtain

$$\frac{Jac(f)}{f}\omega_0 - c\frac{Jac(f_1)}{f_1}\omega_0 = h\omega_0$$

Taking residues at 0 we obtain that

$$n_1 + \cdots + n_r - c(m_1 + \cdots + m_r) = 0$$

where the n_i and m_j are the vanishing orders of the functions $f \circ \nu$ and $f_1 \circ \nu$ at $\nu^{-1}(0)$, respectively. Hence c is a positive rational number. □

The real valued bilinear forms on $\mathbf{A}$ and on $\mathbf{B}_0$ depended on the choice of real valued linear functions $L_{\mathbf{A}} : \mathbf{A} \longrightarrow \mathbb{R}$ and $L_{\mathbf{B}_0} : \mathbf{B}_0 \longrightarrow \mathbb{R}$ which have the property of sending the corresponding Jacobians to a positive number. Having chosen $\mathcal{L}$ and hence $L_{\mathbf{B}_0}$, we will choose $L_{\mathbf{A}}$ subject to the compatibility condition

$$L_{\mathbf{B}_0}|_{\tilde{K}_1} = L_{\mathbf{A}} \circ \varphi. \tag{27}$$

Proof of Theorem 0.3 Let $m \geq 1$ and consider the commutative diagram:

$$\begin{array}{ccccccc}
(f^m : f_1) \oplus (f^m : f_1) & \xrightarrow{\frac{f_1 \cdot}{f^m}\cdot} & (f^m : f_1) & \xrightarrow{\tilde{\pi}_0} & \tilde{K}_1 & \xrightarrow{L_{B_0}} & \mathbb{R} \\
\Phi \oplus \Phi \downarrow & & \downarrow \Phi & & \downarrow \varphi & & \downarrow Id. \\
(f_1 : f) \cap (f^{m-1}) \oplus (f_1 : f) \cap (f^{m-1}) & \xrightarrow{\frac{1\cdot}{f^{m-1}}\cdot} & (f_1 : f) \cap (f^{m-1}) & \xrightarrow{\pi_0} & K_1 & \xrightarrow{L_{\mathbf{A}}} & \mathbb{R}
\end{array}$$

Here the mapping $\frac{f_1\cdot}{f^m}\cdot$ acts on a couple

$$(a, b) \in (f^m : f_1) \oplus (f^m : f_1) \quad \text{by} \quad (a, b) \longrightarrow \frac{af_1}{f^m}b,$$

and similarly for $\frac{1\cdot}{f^{m-1}}\cdot$. The mapping π_0 is obtained by reducing mod (f_1). The vertical maps are isomorphisms, so we may interpret the commutative diagram as providing a conjugation of the top bilinear form into the bottom bilinear form. We reduce the first row by (f) and the second row by (f_1). This is possible since $\Phi(f) = f_1$ and both bilinear forms degenerate in the submodules in the denominator of the quotient. We thus obtain that the m^{th} Jantzen's bilinear form is being conjugated by $\varphi : \tilde{K}_m \longrightarrow K_m$ to the bilinear form $\langle , \rangle_{L_{\mathbf{A}}, f, m}$. □

4 The index of contact vector fields

4.1 The GSV-index $Ind_{V_0,\pm}(X|V_0)$ and the signature function $Sgn_{f,0}(X)$

Let $f : \mathbb{R}^{2N+1} \to \mathbb{R}$ be a germ of a real analytic function with an algebraically isolated singularity at 0. Denote also by f its extension to a germ in $\mathbb{C}^{2N+1}$ and let V_t and $V_t^{\mathbb{C}}$ be the germs of real or complex analytic varieties defined by $f = t$. In this section we prove Theorem 0.4.

We know that both the GSV-index $Ind_{V_0,\pm}(X|V_0)$ and the signature function $Sgn_{f,0}(X)$ verify the law of conservation of numbers (see (29) and similarly for the signature function $Sgn_{f,0}(X)$ [6]). They also coincide in smooth points of the variety. Hence, the two indices differ by a constant K_+ or K_- depending only on the function f (and not on the vector field) and on the positive or negative sign chosen in the GSV-index. Given a function f as above, in order to determine these constants $K_\pm$, it is sufficient to calculate both indices for one vector field X_0 tangent to V_0.

Proof of Theorem 0.4 In order to prove Theorem 0.4, we have to study the index of a family of vector fields tangent to the smoothening $f = t$ of the singular variety $f = 0$. When the ambient space is even dimensional, this was done [7] using the Hamiltonian vector field associated to f. Here, we study the odd-dimensional ambient space $\mathbb{R}^{2N+1}$ and we use the vector fields

$$X_t = (f - t)\frac{\partial}{\partial x_1} + \sum_{i=1}^{N}\left[\frac{\partial f}{\partial x_{2i+1}}\frac{\partial}{\partial x_{2i}} - \frac{\partial f}{\partial x_{2i}}\frac{\partial}{\partial x_{2i+1}}\right],$$

which we call the contact vector fields. The vector field X_t is tangent to V_t, for any t, since $D(f - t)X_t = \frac{\partial f}{\partial x_1}(f - t)$, where $\frac{\partial f}{\partial x_1}$ is the cofactor. For almost all linear hyperplanes through 0 in $\mathbb{C}^{2N+1}$ the projection to this hyperplane gives a description of $V_0^{\mathbb{C}}$ as a branched finite analytic cover [10]. Set $f_j := \frac{\partial f}{\partial x_j}$, with $j = 1, \ldots, 2N + 1$. After perhaps a generic rotation, we may assume that 0 is the only point in its neighborhood that satisfies the equations $f = f_2 = \cdots = f_{2N+1} = 0$, or equivalently such that $f, f_2, \ldots, f_{2N+1}$ is a regular sequence [4]. Hence, the vector field X_0 has an algebraically isolated zero at the origin. The functions $f_1, \ldots, f_{2N+1}$ form a regular sequence, since f has isolated singularities. The hypotheses of the previous part of this paper are satisfied and we apply Sects. 1, 2 and 3 to this situation. Choose a small neighborhood $U_{\mathbb{C}}$ of $0 \in \mathbb{C}^{2N+1}$ and a small $\varepsilon > 0$, as in Sect. 2.1. The derivative of X_t is

$$DX_t := \begin{pmatrix} f_1 & f_2 \quad \cdots \quad f_{2N+1} \\ * & \frac{\partial(f_3, -f_2, \ldots, f_{2N+1}, -f_{2N})}{\partial(x_2, \ldots, x_{2N+1})} \end{pmatrix}. \tag{28}$$

Denote by $Y_t := X_t|_{V_t^{\mathbb{C}}}$ the restriction of X_t to $V_t^{\mathbb{C}}$ or to V_t. The singularities of X_t are always contained in $V_t^{\mathbb{C}}$, and hence X_t and Y_t have the same singularities: $\mathbf{Z}_{\mathbb{C}} \cap V_t^{\mathbb{C}}$.

By definition (see [2,8]), the GSV-index $Ind_{V_\pm,0}(Y_0)$ is the sum of the indices of Y_t at the points $p_t \in V_t$, $\pm t > 0$ small:

$$Ind_{V_0,\pm}(Y_0, 0) = \sum_{\substack{p_t \in U \cap V_t,\, Y_t(p_t) = 0 \\ \pm t > 0}} Ind_{V_t}(Y_t, p_t). \tag{29}$$

Note that V_t is smooth, so the signatures $Ind_{V_t}(Y_t, p_t)$ can be calculated using the usual Eisenbud–Levine, Khimshiashvili formula, on the smooth variety V_t. That is, instead of using the Jacobian $J(X_t)$ as the generator of the socle, one uses the relative Jacobian $J(Y_t)$. In the localization of the algebra $\mathbf{B}_t$ in p_t, we have

$$J(X_t) = f_1 J(Y_t). \tag{30}$$

Hence, the signature of the bilinear form $\langle , \rangle_t^{rel}$ (10) gives the GSV-index:

$$Ind_{V_\pm,0}(X_0) = \tau_\pm. \tag{31}$$

On the other hand, by definition [6], the signature function $Sgn_{f,0}(X)$ is given by the signature of the form $\langle , \rangle_t^{rel}$, for $t = 0$. That is,

$$Sgn_{f,0}(X_0) = \tau_0 \tag{32}$$

It now follows from Jantzen's Proposition 2.1 that the constants $K_\pm$ in Theorem 0.4 are

$$K_+ = \sum_{m\geq 1} \tau_m, \quad K_- = \sum_{m\geq 1} (-1)^m \tau_m. \tag{33}$$

The Theorem 0.4 finally follows from (33) applying Theorem 0.3, which asserts that $\tau_m = \sigma_{A,f,m}$. □

Corollary 2 *Let $\sigma_{\mathbf{A}}$ be the signature of the Jacobian algebra* $\mathbf{A}$ *in* (3)*, and let $\sigma_{\mathbf{A},f,m}$, $m = 0, \ldots, \ell + 1$, be defined as above. Then*

$$\sigma_A = \frac{\chi_+ - \chi_-}{2} = \sum_{m=odd} \sigma_{\mathbf{A},f,m}.$$

Proof By Arnold's formula [1], $2\sigma_{\mathbf{A}}$ equals $\chi_+ - \chi_-$. Now, by the Poincaré-Hopf index theorem, $\chi_+ - \chi_-$ equals $Ind_{V_0,+}(X) - Ind_{V_0,-}(X)$, where X is a real vector field having an algebraically isolated singularity at the origin tangent to V. The Corollary now follows from Theorem 0.4. □

4.2 Examples

Example 4.1 Let f be a quasi-homogeneous real analytic function with an algebraically isolated singularity, i.e. $[f]_{\mathbf{A}} = 0 \in \mathbf{A}$. In this case, $Ann_{\mathbf{A}}(f) = \mathbf{A}$, $M_f = 0$ and $\sigma_{f,1} = \tau_1$ is the only non-zero Jantzen signature of order higher than 0 and it is equal to σ_A. Hence $K_\pm = \chi_\pm = \pm\sigma_{\mathbf{A}}$.

Example 4.2 Let $f = (x^2 + y^3)(x^3 + y^2) + z^2$ and $V = f^{-1}(0) \subset \mathbb{R}^3$. This example is not a quasi-homogeneous singularity. All calculations have been done using the Computer algebra system Singular [9]. The local algebra $\mathbf{A} = \frac{\mathcal{A}_{\mathbb{R}^3,0}}{(f_x, f_y, f_z)}$ has dimension 11, $Ann_{\mathbf{A}}(f)$ is the maximal ideal of dimension 10 and $([f]_{\mathbf{A}})$ is the 1-dimensional socle ideal. We thus have that M_f has nine one-dimensional Jordan blocks and one two-dimensional Jordan block. The Hessian $[Hess(f)]_{\mathbf{A}}$ generating the socle equals $-220[f]_{\mathbf{A}}$ in $\mathbf{A}$. The filtration (4) is given by

$$(f) \subset Ann_{\mathbf{A}}(f) \subset \mathbf{A}.$$

The signature σ_1 is the signature of $\langle , \rangle_{L_{\mathbf{A}}}$ on the 9 dimensional space isomorphic to $\frac{Ann_{\mathbf{A}}(f)}{(f)}$. This signature is equal to 1. The signature σ_2 is given by the the sign of $L_{\mathbf{A}}(\frac{f \cdot f}{f}) = L_{\mathbf{A}}(f) < 0$, so $\sigma_2 = -1$. This gives by Theorem 2 that $K_+ = 0$, $K_- = -2$.

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