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Classical and nonclassical symmetries for a Kuramoto–Sivashinsky equation with dispersive effects

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SUMMARY

We apply the Lie-group formalism and the nonclassical method due to Bluman and Cole to deduce symmetries of the generalized Kuramoto–Sivashinsky equation with dispersive effects. We make a full analysis of the symmetry reductions and we prove that the nonclassical method applied to the equation leads to new reductions, which cannot be obtained by Lie classical symmetries. Some new solutions can be derived. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: partial differential equation; symmetries

1. INTRODUCTION

The Kuramoto–Sivashinsky (KS) equation has been studied by many authors [1–3]. There have been several generalizations of the KS equation such as the generalized KS (GKS) with dispersive effects:

$$u_t + f(u)_x + \alpha u_{xx} + \phi(u)_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = g(u) \tag{1}$$

where α, β and γ are constants, $f(u)$, $\phi(u)$ and $g(u)$ are functions.

In order to obtain analytical solutions, several methods have been applied for some particular cases of (1). In this sense, for $f(u) = u^2/2$, ϕ constant and $g = 0$, Equation (1) is KS equation, and in [1] exact travelling wave solutions are obtained using trigonometric functions expansions method. In [4, 5], the authors have analysed the classical and nonclassical symmetries of Equation (1) with $\gamma = k$, $\alpha = \beta = 0$, $g(u) = 0$, $f(u)$ constant and $\phi_u = F(u)$. In this case, Equation (1) is known as

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the Cahn–Hilliard equation and describe diffusion for decomposition of a one-dimensional binary solution.

If $\gamma=1$, $g(u)=0$, $\phi(u)=(\alpha_2/2)u^2+(\alpha_3-\alpha)u+\alpha_4$ and $f(u)=(\alpha_1/2)u^2+\beta u+\alpha_5$, with α_i constants these equations include the Korteweg-de Vries equations supplements by additional terms of the KS and describing nonlinear convection and the input of energy produced by Marangoni forces on the long scales together with energy dissipation on short scales. In [6], the authors have obtained classical and nonclassical symmetries of the equation and have reduced the equation to ordinary differential equations (ODEs).

The application of Lie transformations group theory for the construction of solutions of nonlinear partial differential equations (PDEs) is one of the most active fields of research in the theory of nonlinear PDEs and applications. Classical and nonclassical symmetries of nonlinear PDEs may be used to reduce the number of independent variables of the PDEs; in particular, we might reduce the PDEs to ODEs. The ODEs may also have symmetries that allow us to reduce the order of the equation, and we can integrate to find exact solutions.

Motivated by the fact that symmetry reductions for many PDEs are unobtainable by using classical symmetries, there have been several generalizations of the classical Lie group method for symmetry reductions. The notion of nonclassical symmetries was firstly introduced by Bluman and Cole [7] to study the symmetry reductions of the heat equation. The description of the method can be found in [8, 9]. Clarkson and Mansfield [10] proposed an algorithm for calculating the determining equations associated with the nonclassical method. Bîlă and Niesen [11] proposed a new procedure for finding nonclassical symmetries. In [12, 13], we extended the procedure described in [11] to a different case.

In this paper, we make a full analysis of the symmetry reductions of Equation (1) with $\gamma=1$. By using the algorithm described in [12, 13], we prove that the nonclassical method applied to the GKS equation (1) leads to new reductions.

2. CLASSICAL SYMMETRIES OF GKS EQUATION

We consider the classical Lie group symmetry analysis of Equation (1). Invariance of Equation (1) under a Lie group of point transformations, with infinitesimal generator

$$V=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u} \tag{2}$$

leads to a set of 10 determining equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$, using the MACSYMA program symmgrp.max [14]. Solving this system we obtain

$$\xi=\frac{x}{4}\tau_t+\delta(t),\quad \tau=\tau(t),\quad \eta=\left(-\frac{\beta x}{16}\tau_t+\theta(t)\right)u+\varphi(x,t)$$

where τ, δ, θ and φ are related by the following conditions:

$$\beta\tau_t\phi_{uuu}ux+\beta\tau_t\phi_{uu}x-16\phi_{uuu}\theta u-16\phi_{uuu}\varphi-16\phi_{uu}\theta-8\tau_t\phi_{uu}=0$$

$$\beta\tau_t\phi_{uu}ux-16\phi_{uu}\theta u-16\phi_{uu}\varphi-8\tau_t\phi_u+3\beta^2\tau_t-8\alpha\tau_t=0$$

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$$\beta\tau_t g_u ux - \beta\tau_{tt} ux - \beta\tau_t gx + 16\theta_t u - 16 g_u \theta u - \beta\tau_t f_u u + 16\varphi_{xxxx} + 16\beta\varphi_{xxx}$$
$$+16\phi_u\varphi_{xx} + 16\alpha\varphi_{xx} + 16 f_u\varphi_x + 16\varphi_t - 16 g_u\varphi + 16 g\theta - 16\tau_t g = 0$$

$$\beta\tau_t f_{uu} ux + 4\tau_{tt} x - 16 f_{uu}\theta u + 2\beta\tau_t\phi_{uu} u - 32\phi_{uu}\varphi_x - 16 f_{uu}\varphi + 16\delta_t + 2\beta\tau_t\phi_u$$
$$-12\tau_t f_u + 2\alpha\beta\tau_t = 0$$

The solutions of this system depend on α, β, $f(u)$, $\phi(u)$ and $g(u)$. For α, β, $f(u)$, $\phi(u)$ and $g(u)$ arbitrary, the only symmetries admitted by (1) are the group of space and time translations, which are defined by the infinitesimal generators:

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}$$

The functional forms of $f(u)$, $\phi(u)$ and $g(u)$ and the constants α, β, for which Equation (1) has the following extra symmetries.

Case 1: If $f = \alpha_2 \exp((3\alpha_1/2)u) + \alpha_3 u + \alpha_4$, $\phi = \alpha_5 \exp(\alpha_1 u) - \alpha u + \alpha_6$, $g = \alpha_7 \exp(2\alpha_1 u)$ and $\beta = 0$,

$$V_3^1 = (x + 3\alpha_3 t)\partial_x + 4t\partial_t - \frac{2}{\alpha_1}\partial_u$$

We observe that for $\alpha = \alpha_2 = \alpha_3 = \alpha_6 = \alpha_7 = 0$, V_3^1 is symmetry of the Cahn–Hilliard equation, obtained in [4, 5].

Case 2: If $f = (u+1)\ln(u+1) + \alpha_1 u + \alpha_2$, $\phi = \alpha_3 u + \alpha_4$ and $g = (u+1)[\alpha_5 \ln(u+1) + \alpha_6]$

- For $\alpha_5 = 1$:

$$V_3^2 = \mathrm{e}^t\partial_x + \mathrm{e}^t(u+1)\partial_u$$

- For $\alpha_5 = 0$:

$$V_3^3 = t\partial_x + (u+1)\partial_u$$

Case 3: If $f = \alpha_0(\alpha_1 u + \alpha_2)^{\frac{3n}{2}+1} + \alpha_3 u + \alpha_4$, $\phi = \alpha_7(\alpha_1 u + \alpha_2)^m + \alpha_5 u + \alpha_6$ and $g = (\alpha_8 u + \alpha_9)^k$ the new symmetries are:

- For $n \neq 0$, $m = n+1$, $k = 2n+1$, $\beta = 0$, $\alpha_5 = -\alpha$, $\alpha_8 = \alpha_1$ and $\alpha_9 = \alpha_2$:

$$V_3^4 = (x + 3\alpha_3 t)\partial_x + 4t\partial_t - \frac{2}{\alpha_1 n}(\alpha_1 u + \alpha_2)\partial_u$$

  We observe that for $\alpha = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = 0$, V_3^4 is symmetry of the Cahn–Hilliard equation obtained in [4, 5]. For $n = \alpha_1 = 1$ and $\alpha_0 = \alpha_7 = \alpha_8 = \alpha_9 = 0$, V_3^4 is a symmetry of the dissipation-modified Korteweg–de Vries equation, which was obtained in [6].

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- For $n=\frac{2}{3}$, $k=1$, $\alpha_1=1$, $\alpha_2=0$, $\alpha_7=0$ and
  - $\alpha_8\neq 0$:

$$V_3^5=\mathrm{e}^{\alpha_8 t}\partial_x+\frac{\alpha_8}{2\alpha_0}\,\mathrm{e}^{\alpha_8 t}\partial_u$$

  - $\alpha_8=0$:

$$V_3^6=t\partial_x+\frac{1}{2\alpha_0}\partial_u$$

  - $\alpha_8=0$, $\beta=0$, $\alpha_5=-\alpha$:

$$V_3^7=V_3^6=t\partial_x+\frac{1}{2\alpha_0}\partial_u,\quad V_4^1=\left(\frac{x}{4}+\frac{7}{4}\alpha_0\alpha_9 t^2\right)\partial_x+t\partial_t+\left(-\frac{3}{4}u+\frac{7}{4}\alpha_9 t-\frac{3\alpha_3}{8\alpha_0}\right)\partial_u$$

- For $k=1$, $\alpha_0=0$ and $\alpha_7=0$,
  - If $\alpha_9=0$:

$$V_3^8=u\partial_u,\quad V_4^2=\omega(x,t)\partial_u$$

    where $\omega(x,t)$ must satisfy equation $\omega_{xxxx}+\beta\omega_{xxx}+(\alpha+\alpha_5)\omega_{xx}+\alpha_3\omega_x+\omega_t-\alpha_8\omega+\alpha_9=0$.
  - If $\alpha_5=(\frac{3}{8}\beta^2-\alpha)$ and $\alpha_9=0$:

$$V_3^9=V_3^8=u\partial_u$$

$$V_4^3=\left[\frac{x}{4}+\left(\frac{3}{4}\alpha_3-\frac{3}{64}\beta^3\right)t\right]\partial_x+t\partial_t+\left[-\frac{\beta}{16}x+\left(\frac{\alpha_3}{16}\beta+\alpha_8\right)t\right]u\partial_u$$

$$V_5=\varpi(x,t)\partial_u$$

    where $\varpi(x,t)$ must satisfy equation $\alpha_9\beta k_2 x-\alpha_3\alpha_9\beta k_2 t-16\alpha_8\alpha_9 k_2 t-16\varpi_{xxxx}-16\beta\varpi_{xxx}-6\beta^2\varpi_{xx}-16\alpha_3\varpi_x-16\varpi_t+16\alpha_8\varpi-16\alpha_9 k_1+16\alpha_9 k_2=0$, with k_1 and k_2 arbitrary constants.

For the sake of completeness, we next provide the generators of the nontrivial one-dimensional optimal system:

For α, β, $f(u)$, $\phi(u)$ and $g(u)$ arbitrary,

$$\langle \mathrm{V}_1\rangle,\ \langle \lambda \mathrm{V}_1+\mathrm{V}_2\rangle,\quad \lambda\in\mathscr{R}$$

The new subalgebras, of the nontrivial one-dimensional optimal system, which are obtained for the different cases, are:

Case 1:

$$\{\langle \mathrm{V}_3^1\rangle\}$$

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Table I. The similarity solutions.

j	U_j	z_j	u_j
1	$\lambda V_1 + V_2$	$x - \lambda t$	$h(z)$
2	V_3^1	$(x - \alpha_3 t)t^{-\frac{1}{4}}$	$h(z) - \ln(t^{\frac{1}{2\alpha_1}})$
3	V_3^2	t	$e^x h(z) - 1$
4	V_3^3	t	$e^{\frac{x}{t}} h(z) - 1$
5	V_3^4	$(x - \alpha_3 t)t^{-\frac{1}{4}}$	$h(z)t^{-\frac{1}{2n}} - \frac{\alpha_2}{\alpha_1}$
6	V_3^6	t	$\frac{x}{2\alpha_0 t} + h(z)$
7	$\lambda V_1 + V_3^5$	t	$\frac{\alpha_8 e^{b_1 t} x}{2\alpha_0(e^{\alpha_8 t}+\lambda)} + h(z)$
8	$\lambda V_1 + V_4^3$	$\frac{4\lambda}{\sqrt[4]{t}} + \frac{x}{\sqrt[4]{t}} + \frac{\beta^3 t^{3/4}}{16} - \alpha_3 t^{3/4}$	$t^{\frac{\beta\lambda}{4}} h e^{-\frac{64\beta\sqrt[4]{t}z+(-\beta^4-256\alpha_8)t}{256}}$
9	$\lambda V_2 + V_4^1$	$\frac{32\alpha_9\lambda^2+8\alpha_9 t\lambda+3x-3\alpha_9 t^2}{3\sqrt{2}\sqrt[4]{\lambda+t}}$	$\frac{-(\lambda+t)^{3/4}(8\sqrt{2}\alpha_9\lambda-6\sqrt{2}\alpha_9 t+3\sqrt{2}\alpha_3)-3h}{6\sqrt{2}(\lambda+t)^{3/4}}$
10	$\lambda V_1 + \mu V_2 + V_3^8$	$\mu x - \lambda t$	$e^{\frac{t}{\mu}} h(z)$

Table II. ODEs.

j	ODE_j
1	$h'''' + \phi_{hh}(h')^2 + \phi_h h'' - \lambda h' + \beta h''' + \alpha h'' + f_h h' - g = 0$
2	$h'''' + \alpha_1(\frac{3}{4}\alpha_3 - \frac{3}{64}\beta^3)e^{\alpha_1 h}h'' + \alpha_1^2(\frac{3}{4}\alpha_3 - \frac{3}{64}\beta^3)e^{\alpha_1 h}(h')^2 + \frac{3\alpha_1}{2}(\frac{\alpha_3}{16}\beta + \alpha_8)e^{\frac{3\alpha_1 h}{2}}h'$ $-\frac{z}{4}h' - \alpha_7 e^{2\alpha_1 h} - \frac{1}{2\alpha_1} = 0$
3	$h' + (\beta + \alpha - \alpha_6 + \alpha_3 + \alpha_1 + 2)h = 0$
4	$z^4 h' + z^3 h\ln(h) + (-\alpha_6 z^4 + (\alpha_1 + 1)z^3 + (\alpha + \alpha_3)z^2 + \beta z + 1)h = 0$
5	$h'''' + \alpha_1^{n+1}\alpha_7(n+1)h^n h'' + \alpha_1^{n+1}\alpha_7(n^2 + n)h^{n-1}(h')^2 - \frac{z}{4}h' + \alpha_0(\frac{3n}{2} + 1)\alpha_1^{\frac{3n+2}{2}} h^{\frac{3n}{2}} h'$ $-\alpha_1^{2n+1}h^{2n+1} - \frac{h}{2n} = 0$
6	$zh' + h - \alpha_9 z + \frac{\alpha_3}{2\alpha_0} = 0$
7	$(e^{\alpha_8 z} + \lambda)h' + (\frac{\alpha_3\alpha_8}{2\alpha_0} - \alpha_9)e^{\alpha_8 z} - \alpha_8\lambda h - \alpha_9\lambda = 0$
8	$4h'''' - zh' + \beta\lambda h = 0$
9	$h'''' + 2hh' - zh' - 3h = 0$
10	$\mu^5 h'''' + \beta\mu^4 h''' + (\alpha_5 + \alpha)\mu^3 h'' + (\alpha_3\mu - \lambda)\mu h' + (1 - \alpha_8\mu)h = 0$

Case 2:

$$\{\langle V_3^3\rangle, \langle\lambda V_1 + V_3^2\rangle, \lambda \in \mathscr{R}\}$$

Case 3:

$$\{\langle V_3^4\rangle, \langle V_3^6\rangle, \langle\lambda V_1 + V_3^5\rangle, \langle\lambda V_1 + V_4^3\rangle, \langle\lambda V_2 + V_4^1\rangle, \langle\lambda V_1 + \mu V_2 + V_3^8\rangle, \lambda, \mu \in \mathscr{R}\}$$

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Having determined the infinitesimals, the similarity solutions z_j and u_j, which are found by solving the invariant surface conditions:

$$\xi(x,t,u)u_x + \tau(x,t,u)u_t - \eta(x,t,u) = 0 \tag{3}$$

are listed in Table I.

In Table II we list the ODEs to which (1) is reduced.

3. NONCLASSICAL SYMMETRIES OF GKS EQUATION

To obtain nonclassical symmetries of (1) for $\tau \neq 0$, we apply the algorithm described in [11]. If $\tau \neq 0$, without loss of generality, we may set $\tau = 1$. The equivalent form of (1), by reducing the initial system using the invariant surface condition $u_t = \eta - \xi u_x$, is

$$u_{xxxx} + \beta u_{xxx} + \mathscr{A}_1(x,t,u)u_{xx} + \mathscr{A}_2(x,t,u)(u_x)^2 + \mathscr{A}_3(x,t,u)u_x + \mathscr{A}_4(x,t,u) = 0 \tag{4}$$

where $\mathscr{A}_1 = \phi_u + \alpha$, $\mathscr{A}_2 = \phi_{uu}$, $\mathscr{A}_3 = f_u - \xi$ and $\mathscr{A}_4 = \eta - g$.

Invariance of (4) under a Lie group of point transformations, with infinitesimal generator (2) leads to the determining equations. Substituting $\tau = 1$ and $\mathscr{A}_i$, with $i = 1, \ldots, 4$, into the determining equations we obtain the determining equations of the nonclassical symmetries of the original PDE Equation (1). The classical method applied to (4) gives only rise to the classical symmetries of (1).

The case for $\tau = 0$ has not been considered in [11]. For $\tau = 0$ we can set $\xi = 1$, without loss of generality. In this case, the invariant surface condition (3) becomes

$$u_x = \eta \tag{5}$$

Substituting (5) into (1) we obtain

$$u_t = \mathscr{A}(x,t,u) \tag{6}$$

where

$$\begin{aligned} \mathscr{A} = & -\eta_{xxx} - \eta_u\eta_{xx} - 3\eta\eta_{uu}\eta_x - 3\eta_{ux}\eta_x - (\eta_u)^2\eta_x - \eta^3\eta_{uuu} - 3\eta^2\eta_{uux} - 4\eta^2\eta_u\eta_{uu} \\ & -3\eta\eta_{uxx} - 5\eta\eta_u\eta_{ux} - \eta(\eta_u)^3 - \beta\eta_{xx} - \beta\eta_u\eta_x - \phi_u\eta_x - \alpha\eta_x - \beta\eta^2\eta_{uu} - 2\beta\eta\eta_{ux} \\ & -\beta\eta(\eta_u)^2 - \phi_u\eta\eta_u - \alpha\eta\eta_u - \phi_{uu}\eta^2 - f_u\eta + g \end{aligned} \tag{7}$$

Invariance of Equation (6) under a Lie group of point transformations with infinitesimal generator (2) leads to two determining equations:

$$\begin{aligned} \mathscr{A}\xi_u + \xi_t &= 0 \\ \mathscr{A}\eta_u + \eta_t - \mathscr{A}_u\eta - \mathscr{A}^2\tau_u - \mathscr{A}\tau_t - \mathscr{A}_t\tau - \mathscr{A}_x\xi &= 0 \end{aligned} \tag{8}$$

Substituting $\xi = 1$ and $\tau = 0$ in (8), we obtain the equation

$$\mathscr{A}\eta_u + \eta_t - \mathscr{A}_u\eta - \mathscr{A}_x = 0 \tag{9}$$

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Table III. The constants, functions and infinitesimals.

i		$\phi_i(u)$	$f_i(u)$	$g_i(u)$	$\eta_i(x,t,u)$
1	$k_6 \neq 0$	$k_6u^3+k_5u+k_4$	$k_3u^2+k_2$	k_1u	$\frac{u}{x+x_0}$
2	$k_6 \neq 0$	$k_6\ln u-\alpha u+k_4$	$u\ln u+k_3u+k_2$	k_1u	$\frac{u}{t+t_0}$
3	$k_6 \neq 0$	$k_6\ln u+k_5u+k_4$	k_3u+k_2	$u(\ln u+k_1)$	$e^t u$
4	$k_6 \neq 0$	$k_6u^2+k_5u+k_4$	$-6k_6k_3u+k_2$	k_1	$\frac{x}{6k_6(t+t_0)}+k_3$
5	$k_3 \neq 0$	$k_6u^2+k_5u+k_4$	$k_3u^2+k_2u+k_1$	0	$\frac{1}{2k_3t}$

Table IV. ODEs.

i	z	u_i	ODE'_i
1	t	$(x+x_0)h(z)$	$h'+6k_6h^3+2k_3h^2-k_1h=0$
2	t	$h(z)e^{\frac{x}{t+t_0}}$	$(z+t_0)^4h'+(z+t_0)^3h\log h+(-k_1t_0^4+(-4k_1z+k_3+1)t_0^3$ $+(3(k_3+1)z-6k_1z^2)t_0^2+(-4k_1z^3+3(k_3+1)z^2+\beta)t_0$ $-k_1z^4+(k_3+1)z^3+\beta z+1)h=0$
3	t	$h(z)e^{e^t x}$	$he^{4z}+\beta he^{3z}+k_5he^{2z}+\alpha he^{2z}+k_3he^z+h'-h\log h-k_1h=0$
4	t	$\frac{x^2}{12k_6(t_0+t)}+k_3x+h(z)$	$6k_6(z+t_0)h'-6k_6(4k_3^2k_6+k_1)(z+t_0)+2k_6h$ $+k_5+\alpha=0$
5	t	$\frac{x}{2k_3t}+h(z)$	$2k_3^2z^2h'+2k_3^2zh+k_2k_3z+k_6=0$

Substituting (7) in (9), we obtain the determining equation for the infinitesimal η:

$$\begin{aligned}&\eta_{xxxx}+4\eta\eta_{uu}\eta_{xx}+4\eta_{ux}\eta_{xx}+3\eta_{uu}(\eta_x)^2+6\eta^2\eta_{uuu}\eta_x+12\eta\eta_{uux}\eta_x+10\eta\eta_u\eta_{uu}\eta_x+6\eta_{uxx}\eta_x\\&\quad+4\eta_u\eta_{ux}\eta_x+\eta^4\eta_{uuuu}+4\eta^3\eta_{uuux}+6\eta^3\eta_u\eta_{uuu}+6\eta^2\eta_{uuxx}+12\eta^2\eta_u\eta_{uux}+4\eta^3(\eta_{uu})^2\\&\quad+12\eta^2\eta_{ux}\eta_{uu}+7\eta^2(\eta_u)^2\eta_{uu}+4\eta\eta_{uxxx}+6\eta\eta_u\eta_{uxx}+8\eta(\eta_{ux})^2+4\eta(\eta_u)^2\eta_{ux}+\beta\eta_{xxx}+\phi_u\eta_{xx}\\&\quad+\alpha\eta_{xx}+3\beta\eta\eta_{uu}\eta_x+3\beta\eta_{ux}\eta_x+3\phi_{uu}\eta\eta_x+f_u\eta_x+\beta\eta^3\eta_{uuu}+3\beta\eta^2\eta_{uux}+3\beta\eta^2\eta_u\eta_{uu}+\phi_u\eta^2\eta_{uu}\\&\quad+\alpha\eta^2\eta_{uu}+3\beta\eta\eta_{uxx}+3\beta\eta\eta_u\eta_{ux}+2\phi_u\eta\eta_{ux}+2\alpha\eta\eta_{ux}+2\phi_{uu}\eta^2\eta_u+g\eta_u+\eta_t+\phi_{uuu}\eta^3+f_{uu}\eta^2\\&\quad-g_u\eta=0\end{aligned} \tag{10}$$

The complexity of this equation is the reason why we cannot solve (10) in general, Thus, we proceed, by making ansatz on the form of $\eta(x,t,u)$, to solve (10). In Table III, we list the functional forms of $f(u)$, $g(u)$ and $\phi(u)$ for which we obtain nonclassical symmetries and the corresponding infinitesimals.

In Table IV we list the transformations and ODEs (ODE'_i) to which (1) is reduced.

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4. ANALYSIS OF SOME REDUCED EQUATIONS

Integrating ODE_i, and ODE'_i we obtain the function $h(z)$ and, substituting into the similarity solution u_j, we obtain the following exact solutions of (1):

- ODE_1, for $\beta=\alpha=f'=g=0$, can be reduced to

$$h''' + \phi' h' - \lambda h = k_1 \tag{11}$$

  Equation (11) admits the generator $X=\mathrm{d}/\mathrm{d}z$. Making the change of variables $y=h$, $w=z$ and, denoting $\delta=\mathrm{d}w/\mathrm{d}y$, Equation (11) is reduced to

$$\delta\frac{\mathrm{d}^2\delta}{\mathrm{d}h^2} - 3\left(\frac{\mathrm{d}\delta}{\mathrm{d}h}\right)^2 + (\lambda h + k_1)\delta^5 + \phi'(h)\delta^4 = 0$$

- For $\phi'=h$, $f=\lambda h + a$ and $\beta=g=0$, integrating ODE_1 twice with respect to z it can be reduced to the following ODE:

$$h'' + \tfrac{1}{2}h^2 + \alpha h = Az + B \tag{12}$$

  By solving Equation (12) for $A=B=0$, we get

$$u(x,t) = -3\alpha\,\mathrm{sech}^2\left[\sqrt{-\frac{\alpha}{4}}(x-\lambda t)\right]$$

- From ODE_3:

$$u(x,t) = k\,\mathrm{e}^{x+(-\beta-\alpha+\alpha_6-\alpha_3-\alpha_1-2)t} - 1$$

- From ODE_4:

$$u(x,t) = \frac{\mathrm{e}^{-\frac{k}{t}-\alpha_1-1}\left(\mathrm{e}^{\frac{x}{t}+\frac{\alpha_6 t}{2}+\frac{\beta}{t^2}+\frac{1}{2t^3}} - t^{\frac{\alpha}{t}+\frac{\alpha_3}{t}}\,\mathrm{e}^{\frac{k}{t}+\alpha_1+1}\right)}{t^{\frac{\alpha}{t}+\frac{\alpha_3}{t}}}$$

- From ODE_6:

$$u(x,t) = \frac{x}{2\alpha_0 t} + \frac{\alpha_9}{2}t + \frac{k}{t} - \frac{\alpha_3}{2\alpha_0}$$

- From ODE_7:

$$u(x,t) = \frac{1}{2\alpha_0(\lambda+\mathrm{e}^{\alpha_8 t})}\Bigg[\Bigg(((2\alpha_0\alpha_9-\alpha_3\alpha_8)t + 2\alpha_0 k - (\log\alpha_0+\log 2)\alpha_3) + 2\alpha_0(\log\alpha_0+\log 2)\alpha_9\Bigg)\mathrm{e}^{\alpha_8 t} - 2\alpha_0\frac{\alpha_9}{\alpha_8}\lambda + \alpha_8\,\mathrm{e}^{\alpha_8 t}x\Bigg]$$

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- From ODE_{10}:

$$u(x,t)=\frac{1}{2\alpha_0\alpha_8(\mathrm{e}^{\alpha_8 t}+\lambda)}((\alpha_8^2 x+(\alpha_8 t+\log(2\alpha_0))(2\alpha_1\alpha_9-\alpha_3\alpha_8)+2\alpha_0\alpha_8 k)\mathrm{e}^{\alpha_8 t}-2\lambda\alpha_0\alpha_9)$$

- From ODE_1', for $6k_1k_6+k_3^2=a^2$, we obtain the solution in implicit form:

$$\left(\frac{6k_6h+k_3-a}{6k_6h+k_3-a}\right)^{k_3}(6k_6h^2+2k_3h-k_1)^a=\mathrm{e}^{-(z+c)2ak_1}h^{2a}$$

- From ODE_1', for $6k_1k_6+k_3^2=-a^2$, we obtain the solution in implicit form:

$$2k_3\arctan\left(\frac{6k_6h+k_3}{a}\right)+\log\left(\frac{6k_6h^2+2k_3h-k_1}{h^2}\right)^a=-2ak_1(z+c)$$

- From ODE_2':

$$u(x,t)=\mathrm{e}^{\frac{x}{t+t_0}+\varphi(t)}$$

with

$$\varphi(t)=\frac{2k_1tt_0^3+(5k_1t^2+(-2k_3-2)t-2k_7)t_0^2+(4k_1t^3+(-4k_3-4)t^2-4k_7t+2\beta)t_0+k_1t^4+(-2k_3-2)t^3-2k_7t^2+2\beta t+1}{2(t+t_0)^3}$$

- From ODE_3':

$$u(x,t)=\mathrm{e}^{\mathrm{e}^t x-\frac{\mathrm{e}^{4t}}{3}-\frac{\beta\,\mathrm{e}^{3t}}{2}-k_5\,\mathrm{e}^{2t}-\alpha\,\mathrm{e}^{2t}-k_3t\,\mathrm{e}^t-k_7\,\mathrm{e}^t-k_1}$$

- From ODE_4':

$$u(x,t)=\frac{x^2}{12k_6(t+t_0)}+k_3x+\frac{k_7}{\sqrt[3]{3}\sqrt[3]{t+t_0}}+3k_3^2k_6(t+t_0)+\frac{3k_1(t+t_0)}{4}-\frac{k_5+\alpha}{2k_6}$$

- From ODE_5':

$$u(x,t)=\frac{k_3x-k_6\log t+2k_3^2k_7}{2k_3^2t}-\frac{k_2}{2k_3}$$

5. CONCLUDING REMARKS

In this paper, the complete Lie-group classification for the GKS equation (1) has been obtained. We have constructed the optimal system and have derived the corresponding reduced equations. In [12] we have extended an algorithm described by Bîlă and Niesen to determine nonclassical symmetries. By using this algorithm, we have derived nonclassical symmetries for Equation (1) that yield to new solutions.

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REFERENCES

1. Fu Z, Liu S. New exact solutions to the KdV–Burguers–Kuramoto equation. *Chaos*, *Solitons and Fractals* 2005; **23**:609–616.
2. Guo BL. Some problems of the generalized Kuramoto–Sivashinsky type equations with dispersive effects. *Research Reports in Physics*. Springer: Berlin, 1990; 236–241.
3. Yang TS. On traveling-wave solutions of the Kuramoto–Sivashinsky equation. *Physica D* 1997; **110**:25–42.
4. Gandarias ML, Bruzón MS. Nonclassical symmetries for a family of Cahn–Hilliard equations. *Physics Letters A* 1999; **263**:331–337.
5. Gandarias ML, Bruzón MS. Symmetry analysis and solutions for a family of Cahn–Hilliard equations. *Reports on Mathematical Physics* 2000; **46**(1–2):89–97.
6. Bruzón MS, Gandarias ML. Symmetry reductions for a disipation-modified KdV equation. *Applied Mathematical Letters* 2003; **16**:155–159.
7. Bluman GW, Cole JD. The general similarity solutions of the heat equation. *Journal of Mathematics and Mechanics* 1969; **18**:1025–1042.
8. Clarkson PA. Nonclassical symmetry reductions of the Boussinesq equation. *Chaos*, *Solitons*, *Fractals* 1995; **5**:2261–2301.
9. Levi D, Winternitz P. Nonclassical symmetry reduction: example of the Boussinesq equation. *Journal of Physics A* 1989; **22**:2915–2924.
10. Clarkson PA, Mansfield EL. Algorithms for the nonclassical method of symmetry reductions. *SIAM Journal on Applied Mathematics* 1994; **55**:1693–1719.
11. Bîlă N, Niesen J. On a new procedure for finding nonclassical symmetries. *Journal of Symbolic Computation* 2004; **38**:1523–1533.
12. Bruzón MS, Gandarias ML. Applying a new algorithm to derive nonclassical symmetries. *Communications in Nonlinear Science and Numerical Simulation*, 2006, in press.
13. Bruzón MS, Gandarias ML. Nonclassical symmetries: applying a new procedure. *Transactions of Nonlinear Science and Complexity*. World Scientific: Singapore, 2006; 7–13.
14. Champagne B, Hereman W, Winternitz P. The computer calculation of Lie point symmetries of large systems of differential equations. *Computer Physics Communications* 1991; **66**:319–340.

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