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# Bounded universal functions in one and several complex variables

**Pamela Gorkin · Fernando León-Saavedra · Raymond Mortini**

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**Abstract** We show how to obtain functions that are universal for the ball of $H^\infty(\Omega)$, where $\Omega \subset \mathbb{C}^n$. The existence of our functions will follow from universality criteria, but we also show how to construct them. Then we study the connection between certain interpolating sequences, runaway automorphisms, and the existence of universal functions on domains in $\mathbb{C}^n$.

**Keywords** Universal functions · Composition operators · Bounded analytic functions · Asymptotic interpolating sequences · Automorphisms of domains in $\mathbb{C}^n$

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## 0 Introduction

The study of universal functions began in 1929 when Birkhoff [4] proved that there exists an entire function $f$ such that the set of translates of $f$ is dense, with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}$, in the space of entire functions. Universal functions have been studied in various contexts, but an underlying common theme is that the domain has a sequence of automorphisms $(\phi_n)$ and these automorphisms have the property that there exists a function $f$ such that $\{f \circ \phi_n : n \in \mathbb{N}\}$ is dense in an appropriate

P. Gorkin
Department of Mathematics, Bucknell University, Lewisburg, PA 17837, USA
e-mail: pgorkin@bucknell.edu

F. León-Saavedra
Departamento de Matemáticas, University of Cádiz, 11402-Jerez de la Frontera, Spain
e-mail: fernando.leon@uca.es

R. Mortini (✉)
Département de Mathématiques, Université Paul Verlaine, Ile du Saulcy, 57045 Metz, France
e-mail: mortini@math.univ-metz.fr

topological space. The function $f$ is called a universal function. Stated in operator theoretic terms, universality looks like the following: given a sequence, $(T_n)$, of bounded linear operators on a topological vector space $X$, a vector $x$ is called universal if the set $\{T_n x : n \in \mathbb{N}\}$ is dense in $X$. The connection between these two concepts becomes apparent if the operators $T_n$ are chosen to be composition operators $C_{\phi_n}$ where $C_{\phi_n}(f) = f \circ \phi_n$.

Some of the existence theorems obtained thus far are constructive, but they depend on properties particular to the domain (see, for example, [3,5,11,15,20,21]). Most results are nonconstructive and the existence of a universal vector comes about because of a universality theorem based on a Baire-type argument. Gethner and Shapiro's paper [9] is not constructive; the importance of their paper comes from the fact that it unifies many of the existing theorems. We will refer to theorems of the type discovered by Gethner and Shapiro as universality criteria.

The setting in this paper is the following. We let $\Omega \subseteq \mathbb{C}^N$ be a domain, and denote by $H(\Omega)$ the space of holomorphic functions on $\Omega$. Throughout this paper we assume that $H^\infty(\Omega)$, the space of all bounded holomorphic functions on $\Omega$, is not trivial. We consider universal functions for $H(\Omega)$ relative to self-maps, $\phi_n$, of $\Omega$, as well as universal functions for the closed unit ball $\mathscr{B} = \{f \in H^\infty(\Omega) : ||f||_\infty \leq 1\}$ of $H^\infty(\Omega)$: a function $f \in \mathscr{B}$ is universal for $\mathscr{B}$ or $\mathscr{B}$-universal with respect to $(\phi_n)$ if $\{f \circ \phi_n : n \in \mathbb{N}\}$ is dense in $\mathscr{B}$ in the compact open topology. In this paper, the first two sections contain preliminary information as well as the universality theorems that we will need. In Sect. 3, we will establish the existence of $\mathscr{B}$-universal functions as a consequence of our universality criteria. In Sect. 4, we will show how to construct such universal functions. For domains in which peak functions are easily obtainable, this construction is quite explicit. We will also relate our results to the current state of affairs.

Thus far, proofs of existence of universal functions for $H(\Omega)$ have focused on a notion of convergence to the boundary, which we now describe. A sequence of automorphisms, $(\phi_j)$, is said to be runaway ([3,20], [2, p. 24]) if for every compact set $K \subset \Omega$, there exists a natural number $n = n(K)$ such that $K \cap \phi_n(K) = \emptyset$. Working with an exhaustion sequence of compact sets in $\Omega$ and passing to a subsequence, if necessary, we see that if we have a runaway sequence of automorphisms $(\phi_j)$, we can choose a subsequence of $(\phi_j)$ that has the property that for each compact set $K$, there exists an integer $n = n(K)$ such that $\phi_m(K) \cap K = \emptyset$ for all $m \geq n$.

In this paper we show that the existence of $\mathscr{B}$-universal functions is closely related to a certain kind of interpolation: In previous work, Gorkin and Mortini [10] considered asymptotic interpolating sequences of type one; we recall that a sequence $(z_n) \in \Omega^{\mathbb{N}}$ is called an asymptotic interpolating sequence of type one for $H^\infty(\Omega)$ if for every $(a_n) \in \ell^\infty$ with $\sup|a_n| \leq 1$ there exists $f$ in the unit ball $\mathscr{B}$ of $H^\infty(\Omega)$ such that $|f(z_n) - a_n| \to 0$. Sequences $(w_n)$ in $\Omega$ for which there exist $f \in \mathscr{B}$ such that $|f(w_n)| \to 1$ and $|f(w_n)| < 1$ play an important role in interpolation theory. In [10], for example, it was shown that every such sequence contains an asymptotic interpolating sequence of type one for $H^\infty(\Omega)$. Conversely, every asymptotic interpolating sequence $(w_n)$ of type one has the property that there exists a function $f \in H^\infty(\Omega)$ of norm one such that $|f(w_n)| < 1$ and $|f(w_n)| \to 1$ (see [10]). Asymptotic interpolating sequences of type one are always plentiful in spaces in which a point of the boundary is a peak point. In Sect. 4, we shall show that asymptotic interpolating sequences of type one admit not only the possibility of approximating sequences of complex numbers by elements in $\mathscr{B}$ (that is, $|f(z_n) - a_n| \to 0$), but they allow approximation of functions; in other words, there exists a function $F \in H^\infty(\Omega)$ such that for any sequence of bounded analytic functions $h_j \in \mathscr{B}$ we have $|F \circ \phi_{k_j} - h_j \circ \phi_{k_j}| \to 0$ uniformly on compacta, where $\phi_j$ are holomorphic self-maps of $\Omega$ such that $(\phi_j(z_0))_{j \in \mathbb{N}}$ contains an asymptotic interpolating subsequence of type one. We discuss the relationship between runaway sequences of automorphisms and a sequence of automorphisms $(\phi_j)$ such that there is a point $z_0 \in \Omega$ for which $(\phi_j(z_0))$ contains an asymptotic interpolating sequence of type one.

At this point an interesting question is what the existence of a runaway sequence of automorphisms of a domain implies about the domain. We provide some examples of infinitely connected domains that have runaway sequences of automorphisms. Our examples are slight modifications of those that appeared in the recent survey article of Kim and Krantz [18]. The relevant theorems for domains in $\mathbb{C}^N$ are provided in this article as well.

## 1 The universality criterion

Let $X$ be an $F$-space; that is, a complete metrizable topological vector space. A vector $x \in X$ is called a universal vector for the sequence $(T_n)$ of continuous linear operators on $X$ if the set $\{T_n x : n \in \mathbb{N}\}$ is dense in $X$. If $T_n$ is the $n$-th iterate (or $n$-th power) of a single operator, T, then $x$ is called a hypercyclic vector and $T$ is called hypercyclic also.

Let $d$ be a translation invariant distance function inducing the metric topology of the $F$-space $X$. If, in addition, $X$ is locally convex and therefore a Fréchet space, we will also consider a countable system $\{p_n : n \in \mathbb{N}\}$ of seminorms generating the topology of $X$. To simplify our notation, we let $||f|| = d(f, 0)$. As usual, $B_d(y, \delta) = \{x \in X : d(x, y) < \delta\}$ is a $d$-ball. A basic open neighborhood of the origin will be of the form $B = \{x \in X : ||x|| < \varepsilon_0\}$ or $B = \bigcap_{j=1}^{N} \{x \in X : p_j(x) < \varepsilon_j\}$ for some $\varepsilon_j > 0$.

**Definition 1.1** Let $B$ be a basic open neighborhood of the origin in $X$ and let $(T_n)$ be a sequence of continuous linear operators on $X$ such that $B$ is an invariant set for each $T_n$; that is, $T_n(B) \subseteq B$ for $n \in \mathbb{N}$. Then a vector $x \in B$ is called $B$-universal for $(T_n)$ if $\{T_n x : n \in \mathbb{N}\}$ is dense in $B$.

We note that if $x$ is a $B_d(0, \delta)$-universal vector for the sequence of powers of an operator $T$, then $T$ is supercyclic; that is, $\{\lambda T^n : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in $X$.

The following is a version of the classical universality criterion of Gethner and Shapiro ([9, p. 283]) and Kitai [19]. For the reader's convenience, we present the proof, which is a slight modification of that in [9].

**Theorem 1.2** (Universality criterion, first version) *Let $X$ be a separable $F$-space, $B$ a basic open neighborhood of the origin in $X$, $M$ a closed subset of $X$ and $(T_n)$ a sequence of continuous linear operators on $X$. Suppose that for each $n$ the operator $T_n$ has a right inverse $S_n$ (not necessarily linear or continuous). In addition, suppose that there are two sets $D_1$ and $D_2$, $D_1$ dense in $B$, $D_2$ dense in $M$, such that $T_n \to 0$ on $D_1$ and $S_n \to 0$ on $D_2$. Then there exists a $G_\delta$-set $\mathcal{U}$ that is dense in $B$, consisting of vectors $x$ such that $M$ is contained in the closure of $\{T_n x : n \in \mathbb{N}\}$.*

We note that this theorem yields the existence of sets $G_n$, that are open in $X$, such that $\mathcal{U} = \bigcap(G_n \cap B)$.

*Proof* In case $X$ is a Fréchet space, we let $|||\cdot|||$ be one of the seminorms $\{p_j : j \in \mathbb{N}\}$ used to generate the topology of $X$. Without loss of generality, we may assume that $B = \{x \in X : |||x||| < 1\}$. (The cases $B = \bigcap_{j=1}^{N} \{x \in X : p_j(x) < \varepsilon_j\}$ and $B = \{x \in X : ||x|| < 1\}$ work in the same fashion.) Note that, as an open subset of $X$, the set $B$ is a Baire space again. We follow the proof of Gethner and Shapiro [9].

Fix a countable dense subset $\{y_j : j \in \mathbb{N}\}$ of $M$. For $j, N, k \in \mathbb{N}$ let

$$V(j, N, k) = \{x \in B : ||T_n x - y_j|| < 1/k \text{ for some } n \geq N\}.$$

Each of these sets, being a union of sets of the form $B \cap T_n^{-1}(B_d(y_j, 1/k))$, is open in $B$. Since

$$\mathcal{U} = \bigcap_j \bigcap_N \bigcap_k V(j, N, k), \tag{1.1}$$

if we show that each $V(j, N, k)$ is dense in $B$, we may use the Baire category theorem to obtain the desired conclusion.

To establish density, fix $V = V(j, N, k)$. Let $z \in B$. Choose $\delta > 0$ so that $\delta < 1 - |||z|||$. We shall show that $V \cap B_d(z, \delta) \neq \emptyset$. Since the set $D_1$ is dense in $B$ and $D_2$ dense in $M$, we can choose $z_0 \in D_1$ and $y_0 \in D_2$ with $\min\{|||z - z_0|||, ||z - z_0||\} < \delta/2$ and $||y_j - y_0|| < 1/(2k)$. Since the sequences $T_n$ and $S_n$ converge pointwise to 0 on $D_1$ and $D_2$, respectively, we may choose $m \in \mathbb{N}$, $m > N$, such that simultaneously $||T_m z_0|| < 1/(2k)$ and $||S_m y_0|| < \delta/2$. Write $x = S_m y_0 + z_0$. Then $x \in B$, since

$$|||x||| \leq \delta/2 + |||z_0||| \leq \delta/2 + |||z - z_0||| + |||z||| < \delta/2 + \delta/2 + 1 - \delta = 1,$$

and $x \in B_d(z, \delta)$, since

$$d(x, z) = ||x - z|| \leq ||x - z_0|| + ||z - z_0|| = ||S_m y_0|| + ||z - z_0|| < \delta/2 + \delta/2.$$

Moreover, since $T_m S_m$ is the identity map on $X$ we have

$$||T_m x - y_j|| = ||T_m S_m y_0 + T_m z_0 - y_j|| \leq ||y_0 - y_j|| + ||T_m z_0|| < 1/(2k) + 1/(2k) = 1/k.$$

So $x \in V$.

We turn to a variant of the universality criterion. We call a set of the form $\mathcal{U}$, given in Eq. (1.1) above, a $G_\delta$-set of $(B, M)$-universal vectors.

**Theorem 1.3** (Universality criterion, second version) *Let $X$ be a separable $F$-space, $B$ a basic open neighborhood of the origin in $X$, $M$ a closed subset of $X$ and $(T_n)$ a sequence of continuous linear operators on $X$. Let $\{t_{n,m} : n, m \in \mathbb{N}\}$ be a doubly-indexed sequence in $B$ and $\{q_m : m \in \mathbb{N}\}$ a countable dense subset of $M$. Suppose that $T_n$ converges to 0 on a dense subset of $B$ and that for each $m$ we have $t_{n,m} \to 0$ as $n \to \infty$ and $T_n t_{n,m} \to q_m$ as $n \to \infty$. Then there exists a $G_\delta$-set that is dense in $B$, consisting of $(B, M)$-universal vectors for $(T_n)$.*

*Proof* Let $y_{n,m} = T_n t_{n,m}$ and

$$W(N, k, m) = \{x \in B : ||T_n x - y_{n,m}|| < 1/k \text{ for some } n \geq N\}.$$

Then, as before, it can be shown that $W(N, k, m)$ is a dense open subset of $B$ (just write $x = t_{n,m} + z_0$ in place of $S_n y_0 + z_0$). Now the set of $(B, M)$-universal vectors equals the set

$$\bigcap_m \bigcap_N \bigcap_k W(N, k, m),$$

which, by Baire's theorem, is a dense $G_\delta$-subset of $B$.

Of course, Theorems 1.2 and 1.3 are true when $B$ and $M$ are replaced by $X$.

## 2 Universality theorems for $H(\Omega)$ and $H^\infty(\Omega)$

In this section, we consider the case in which the set $X$ in the universality theorem above is the space $H(\Omega)$ of all holomorphic functions on a domain $\Omega \subseteq \mathbb{C}^N$ endowed with the compact open topology. Before we turn to the special universality theorems, we isolate some results that we will use repeatedly.

A translation invariant metric $d$ on $H(\Omega)$ inducing the compact-open topology is given by

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||f - g||_n}{1 + ||f - g||_n}, \tag{2.1}$$

where $||f||_n = \max\{|f(z)| : z \in K_n\}$ for some sequence $(K_n)$ of compact sets satisfying $K_n \subseteq K_{n+1}^\circ$ and $\cup_n K_n = \Omega$. All topological notions in this paper refer to the topology induced by $d$.

We will be interested, primarily, in the space $H^\infty(\Omega)$ of all bounded holomorphic functions on $\Omega$. For $f \in H(\Omega)$ let $\|f\|_\infty = \sup_{z\in\Omega} |f(z)|$ and let $\mathscr{B} = \{f \in H(\Omega) : ||f||_\infty \le 1\}$. Using special versions of the universality criteria above, we shall give a necessary and sufficient condition on the holomorphic maps $\phi_n$ for there to exist $\mathscr{B}$-universal functions relative to the compact open topology. Our results depend on the fact presented below. Since we were unable to locate a reference for it, we include the short proof here.

**Proposition 2.1** *Let $\Omega$ be a domain in $\mathbb{C}^N$. Then the sets $H(\Omega)$ and $H^\infty(\Omega)$ are separable in the compact open topology.*

*Proof* Consider $\Omega$ as a subset of $\mathbb{R}^{2N}$ and $H(\Omega)$ as a subspace of $C(\Omega)$, the space of continuous, complex-valued functions on $\Omega$. Using the Stone Weierstrass theorem, we can approximate a continuous function uniformly on a compact subset $K \subseteq \Omega$ by a polynomial in $2N$ real variables with coefficients in $\mathbb{Q}+i\mathbb{Q}$. By exhausting $\Omega$ with an increasing sequence of compact subsets, we see that $C(\Omega)$ is separable in the compact open topology. Since this topology is given by a metric, the subspaces $H(\Omega)$ and $H^\infty(\Omega)$ are separable as well.

What is new to our approach to universal functions is the concept of asymptotic interpolating sequence of type one introduced in [10].

**Definition 2.2** A sequence $(z_n)$ in a domain $\Omega \subseteq \mathbb{C}^N$ is said to be an asymptotic interpolating sequence of type one if for every sequence $(a_n)$ of complex numbers with $\sup_n |a_n| \le 1$ there exists a function $f$ in the unit ball $\mathscr{B}$ of $H^\infty(\Omega)$ such that

$$|f(z_n) - a_n| \to 0.$$

The following lemma [10, Proposition 4.1], is essential to what follows.

**Lemma 2.3** *Suppose that $H^\infty(\Omega)$ is non-trivial. Let $(z_n)$ be a sequence in $\Omega$. Then $(z_n)$ contains an asymptotic interpolating subsequence of type one if there exists $F \in H^\infty(\Omega)$ such that $||F||_\infty = 1$, $|F(z_n)| < 1$ and $|F(z_n)| \to 1$. Conversely, if $(z_n)$ is an asymptotic interpolating sequence of type one, then there exists such a function $F$.*

Note that any subsequence of an asymptotic interpolating sequence of type one is an asymptotic interpolating sequence of type one, too.

In what follows we let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk and we let $\rho_\Omega(z, w) = \sup\{|f(z)| : f \in \mathscr{B}, f(w) = 0\}$ denote the pseudohyperbolic metric in the algebra $H^\infty(\Omega)$. (The reader is referred to [6] for general properties of the pseudohyperbolic metric.) We note that if $H^\infty(\Omega)$ does not separate points, then $\rho$ is a semimetric. This is the case, for example, if $\Omega = \mathbb{C} \times \mathbb{D}$.

**Lemma 2.4** ([8,10]) *(a) Let $(a, b) \in \mathbb{D} \times \mathbb{D}$ and $(\alpha, \beta) \in \partial\mathbb{D} \times \partial\mathbb{D}$. Then there is a unique Möbius transformation $\psi(z) = e^{i\theta} \dfrac{z_0 - z}{1 - \overline{z}_0 z}$ with $\psi(a) = b$ and $\psi(\alpha) = \beta$.*

*(b) Let $(a_j, b_j) \in \mathbb{D} \times \mathbb{D}$ satisfy $\rho_{\mathbb{D}}(a_1, a_2) = \rho_{\mathbb{D}}(b_1, b_2)$. Then there exists a Möbius transformation $\Psi$ such that $\Psi(a_j) = b_j$ for $j = 1, 2$.*

**Lemma 2.5** *Let $(z_n)$ be an asymptotic interpolating sequence of type one in a domain $\Omega \subseteq \mathbb{C}^N$. Then for any $a \in \Omega$ and $\eta \in \mathbb{D}$ there exists $f \in \mathscr{B}$ such that $f(a) = \eta$ and $f(z_n) \to -1$.*

*Proof* Since $(z_n)$ is an asymptotic interpolating sequence of type one, there exists a nonconstant function $h \in \mathscr{B}$ such that $h(z_{2n}) \to 1$ and $h(z_{2n+1}) \to -1$. Then $g := -h^2$ is a nonconstant function in $\mathscr{B}$ and $g(z_n) \to -1$. By composing on the left with a Möbius transformation $\psi$ that satisfies $\psi(-1) = -1$ and $\psi(g(a)) = \eta$, we obtain the desired function $f = \psi \circ g$.

**Lemma 2.6** *Let $(z_n)$ be a sequence in $\Omega$ such that for some $a \in \Omega$, $\rho_\Omega(z_n, a) \to 1$. Then there exists a sequence $(f_n)$ in $\mathscr{B}$ such that $f_n(z_n) \to -1$ and $f_n(a) \to 1$.*

*Proof* Fix $n$ and choose $g_n \in \mathscr{B}$ so that $g_n(a) = 0$ and $\rho_n' := |g_n(z_n)| > \rho_\Omega(z_n, a)\left(1 - \frac{1}{n}\right)$. By rotating $g_n$ we may assume that $g_n(z_n) = \rho_n'$. Now choose $\eta_n \in ]0, 1[$ such that $\frac{2\eta_n}{1+\eta_n^2} = \rho_n'$. Then $\eta_n \to 1$ and $\rho_n' = \rho_{\mathbb{D}}(\eta_n, -\eta_n)$. By Lemma 2.4, for each $n$, there exists a Möbius transformation $\Psi_n$ such that $\Psi_n(0) = \eta_n$ and $\Psi_n(\rho_n') = -\eta_n$. The function $f_n = \Psi_n \circ g_n$ now satisfies $f_n(a) = \eta_n \to 1$ and $f_n(z_n) = \Psi_n(\rho_n') = -\eta_n \to -1$.

In the following, let $C_{\phi_n}$ be the composition operator associated with the symbol $\phi_n$. We are interested in $\mathscr{B}$-universal vectors for $(\phi_n)$. Note that Theorem 1.2 cannot be applied directly, because $\mathscr{B} = \{f \in H(\Omega) : \sup_{z\in\Omega} |f(z)| \le 1\}$ is not a basic open neighborhood of the origin. One of our tools will be Birkhoff's transitivity criterion (for example see [13, p. 348]).

**Birkhoff's Criterion** *If $(T_n)$ is a sequence of continuous maps on a second-countable Baire space $X$, then there exists a dense set of points $x \in X$ for which $\{T_n(x) : n \in \mathbb{N}\}$ is dense in $X$ if and only if for every pair of open sets $U$ and $V$ in $X$ there exists $n$ such that $T_n(U) \cap V \ne \emptyset$.*

Note that the set $\mathcal{U}$ of such points, called universal elements, is given by $\mathcal{U} = \bigcap_k \bigcup_n T_n^{-1}(U_k)$, where $\{U_k : k \in \mathbb{N}\}$ is a basis for the topology of $X$.

**Theorem 2.7** ($Y$ universality criterion) *Let $(C_{\phi_n})$ be a sequence of invertible composition operators on $H(\Omega)$.*

(a) *Let $Y = H(\Omega)$. Suppose that there are two dense sets $D_1$ and $D_2$ in $Y$ such that $C_{\phi_n} \to 0$ on $D_1$ and $C_{\phi_n}^{-1} \to 0$ on $D_2$. Then there exists a $G_\delta$-set that is dense in $Y$, consisting of $Y$-universal functions for $(\phi_n)$ and $(\phi_n^{-1})$.*
(b) *Let $Y = \mathscr{B}$. Suppose that there are two dense sets $D_1 = \{d_n : n \in \mathbb{N}\}$ and $D_2 = \{\tilde{d}_j : j \in \mathbb{N}\}$ in $\mathscr{B}$ and functions $d_{n,m} \in \mathscr{B}$ with $\lim_m d_{n,m} = 1$, such that $C_{\phi_n} \to 0$ on $D_1$ and $C_{\phi_n}^{-1} \to 0$ on $D_2$. Suppose further that*

$$\{d_n + (\tilde{d}_j d_{n,m}) \circ \phi_m^{-1} : j, m, n \in \mathbb{N}\} \subseteq \mathscr{B}.$$

*Then there exists a $G_\delta$-set that is dense in $\mathscr{B}$, consisting of $\mathscr{B}$-universal functions for $(\phi_n)$ and $(\phi_n^{-1})$.*

We note that the set of universal functions that we find are functions $h \in Y$ for which $\{h \circ \phi_n : n \in \mathbb{N}\}$ is locally uniformly dense in $Y$.

*Proof* Note that both $H(\Omega)$ and $\mathscr{B}$ are Baire spaces with respect to the compact open topology. The case $Y = H(\Omega)$ follows from Theorem 1.2 by setting $B = M = Y$.

Now we consider $Y = \mathscr{B}$. The proof proceeds in the usual way but, in addition, requires Birkhoff's criterion stated above. Recall that $||f||_d = d(f, 0)$, where $d$ is the usual distance (2.1) in $H(\Omega)$ and that $d$ has the property that $||pf||_d \le ||f||_d$ whenever $||p||_\infty \le 1$.

Let $U$ and $V$ be two open sets in $\mathscr{B}$. Since $D_1$ and $D_2$ are dense in $\mathscr{B}$, we may choose $d_n \in U$ and $\tilde{d}_j \in V$. Let $\varepsilon > 0$ be so small that $B_d(\tilde{d}_j, \varepsilon) \cap \mathscr{B} \subseteq V$. Since $C_{\phi_m}$ converges to 0 on $D_1$, there exists $m_0 \in \mathbb{N}$ such that $||d_n \circ \phi_m||_d < \varepsilon/2$ for every $m \ge m_0$. Since $d_{n,m} \to 1$ as $m \to \infty$, there exists $m_1 \ge m_0$ so that $||1 - d_{n,m}||_d < \varepsilon/2$ for every $m \ge m_1$. By our hypothesis,

$$H_{n,m} := d_n + (\tilde{d}_j d_{n,m}) \circ \phi_m^{-1} \in \mathscr{B}.$$

Hence, when $m$ is big enough, $H_{m,n} \in U$ (because $\tilde{d}_j \circ \phi_m^{-1} \to 0$ as $m \to \infty$), and

$$||\tilde{d}_j(d_{n,m} - 1)||_d + ||d_n \circ \phi_m||_d < \varepsilon.$$

So, for some $m$,

$$C_{\phi_m}(H_{n,m}) = d_n \circ \phi_m + \tilde{d}_j d_{n,m} \in B_d(\tilde{d}_j, \varepsilon) \cap \mathscr{B} \subseteq V,$$

because

$$||H_{n,m} \circ \phi_m - \tilde{d}_j||_d \le ||d_n \circ \phi_m + \tilde{d}_j(d_{n,m} - 1)||_d < \varepsilon.$$

Hence $C_{\phi_m}(U) \cap V \ne \emptyset$. Thus we get a dense $G_\delta$-set of $\mathscr{B}$-universal functions for $(\phi_n)$. But $C_{\phi_m}(U) \cap V \ne \emptyset$ implies that $U \cap C_{\phi_m}^{-1}(V) \ne \emptyset$. Hence, by Birkhoff's transitivity criterion, there exists a dense $G_\delta$-set of $\mathscr{B}$-universal function for $(\phi_m^{-1})$. The intersection of both sets yields the dense $G_\delta$-set consisting of $\mathscr{B}$-universal functions for $(\phi_n)$ and $(\phi_n^{-1})$.

## 3 Getting universal functions from universality theorems

Now we apply these universality criteria to establish the existence of $\mathscr{B}$-universal functions in $H(\Omega)$, where $\Omega$ is a domain in $\mathbb{C}^N$. Let us begin with two elementary but useful results.

**Lemma 3.1** *Let $(\phi_n)$ be a sequence of self-maps of $\Omega$. If there exists a $\mathscr{B}$-universal function for $(\phi_n)$, then for any $z_0 \in \Omega$, $(\phi_n(z_0))$ contains an asymptotic interpolating sequence of type one.*

*Proof* Let $F$ denote the $\mathscr{B}$-universal function. Then choose $\phi_{n_j}$ so that $F \circ \phi_{n_j}$ converges to the constant function 1. Evaluation at any $z_0 \in \Omega$ will yield a sequence $(w_j) := (\phi_{n_j}(z_0))_j$ in $\Omega$ for which there exists a nonconstant function $G$ of norm one, namely $F$, such that $G(w_j) \to 1$. By Lemma 2.3, such sequences contain asymptotic interpolating sequences of type one.

**Lemma 3.2** *Let $(\phi_n)$ be a sequence of self-maps of $\Omega$. If there exists $z_0 \in \Omega$ such that $(\phi_n(z_0))$ has an asymptotic interpolating subsequence of type one, then for each $z \in \Omega$ the sequence $(\phi_n(z))$ contains an asymptotic interpolating subsequence of type one, too.*

*Proof* Without loss of generality we may assume that $(\phi_n(z_0))$ is an asymptotic interpolating sequence of type one. By Lemma 2.3, there exists a function $f \in \mathscr{B}$ such that $|f(\phi_n(z_0))| \to 1$ and $|f(\phi_n(z_0))| < 1$ for all $n$. By passing to a subsequence, we may assume that $((f \circ \phi_n)(z_0))$ converges to a point $\alpha$ of modulus one. Thus, $f \circ \phi_n \to \alpha$ uniformly on compact subsets. In particular, $f(\phi_n(z)) \to \alpha$. Again using Lemma 2.3 we see that $(\phi_n(z))$ has an asymptotic interpolating subsequence of type one.

**Lemma 3.3** *Let $(\phi_n)$ be a sequence of automorphisms of $\Omega \subseteq \mathbb{C}^N$. Suppose that for some $z_0 \in \Omega$, $(\phi_n(z_0))$ is an asymptotic interpolating sequence of type one. Then there exist functions $k_n \in \mathscr{B}$ such that*

$$k_n \circ \phi_n \to 1 \text{ as } n \to \infty$$

*and*

$$k_n \circ \phi_n \circ \phi_m \to -1 \text{ as } m \to \infty.$$

*Proof* Since $\phi_j$ is an automorphism and $(\phi_n(z_0))$ is an asymptotic interpolating sequence of type one, we see that $\big((\phi_j \circ \phi_n)(z_0)\big)_n$ is an asymptotic interpolating sequence of type one, too. Hence, by Lemma 2.5, for each $j$ we can find an element $k_j \in \mathscr{B}$ such that $k_j(\phi_j(z_0)) = 1 - 1/j$ and $k_j(\phi_j(\phi_m(z_0))) \to -1$ as $m \to \infty$. Note that since $k_j \in \mathscr{B}$ and $(k_j \circ \phi_j)(z_0) \to 1$, we have $k_j \circ \phi_j \to 1$ uniformly on compacta as $j \to \infty$. Also, since $(k_j \circ \phi_j \circ \phi_m)(z_0) \to -1$ as $m \to \infty$, we have $k_j \circ \phi_j \circ \phi_m \to -1$ uniformly on compacta as $m \to \infty$.

We say that a function $f \in \mathscr{B}$ is a $\mathscr{B}$-biuniversal function for $(\phi_n)$ if $\{f \circ \phi_n : n \in \mathbb{N}\}$ and $\{f \circ \phi_n^{-1} : n \in \mathbb{N}\}$ are locally uniformly dense in $\mathscr{B}$.

**Theorem 3.4** *Let $\Omega \subseteq \mathbb{C}^N$ be a domain for which $H^\infty(\Omega)$ is non-trivial. Let $(\phi_n)$ be a sequence of automorphisms of $\Omega$ and let $z_0 \in \Omega$. Then the following are equivalent.*

(a) *There is a subsequence $(\phi_{n_k})$ of $(\phi_n)$ such that both $(\phi_{n_k}(z_0))$ and $(\phi_{n_k}^{-1}(z_0))$ are asymptotic interpolating sequences of type one.*
(b) *There exist a subsequence $(\phi_{n_k})$ of $(\phi_n)$, two sets $D_1 = \{d_n : n \in \mathbb{N}\}$ and $D_2 = \{\tilde{d}_j : j \in \mathbb{N}\}$ dense in $\mathscr{B}$ and functions $d_{n,m} \in \mathscr{B}$ with $d_{n,m} \to 1$ as $m \to \infty$, such that $C_{\phi_{n_k}} \overset{k\to\infty}{\longrightarrow} 0$ on $D_1$, $C_{\phi_{n_k}}^{-1} \overset{k\to\infty}{\longrightarrow} 0$ on $D_2$ and*

$$\{d_n + \tilde{d}_j d_{n,k} \circ \phi_{n_k}^{-1} : j, n, k \in \mathbb{N}\} \subseteq \mathscr{B}.$$

(c) *There exists a subsequence $(\phi_{n_j})$ of $(\phi_n)$ for which $(\phi_{n_j}(z_0))_j$ is an asymptotic interpolating sequence of type one and a $\mathscr{B}$-biuniversal function $f \in \mathscr{B}$ for $(\phi_n)$ such that the set $\{f \circ \phi_{n_j}^{-1} : j \in \mathbb{N}\}$ is locally uniformly dense in $\mathscr{B}$.*

*Proof* (a) $\Longrightarrow$ (b): Since we know that, with respect to the compact open topology, $H^\infty(\Omega)$ is separable, the subset $\mathscr{B}$ is separable, also. Let $\{p_n : n \in \mathbb{N}\}$ and $\{q_n : n \in \mathbb{N}\}$ be countable dense subsets of $\mathscr{B}$. Now we are assuming that $(\phi_{n_k}(z_0))$ is an asymptotic interpolating sequence of type one (which we denote with the index $m$). Use Lemma 3.3 to obtain functions $k_m \in \mathscr{B}$ satisfying the conditions of that lemma for $(\phi_m(z_0))$. Consider the function $g_j = (1 + k_j)/2$. Then $g_j \circ \phi_j \to 1$ and $g_j \circ \phi_j \circ \phi_m \underset{m\to\infty}{\to} 0$ uniformly on compacta.

Now we also assume that $(\phi_m^{-1}(z_0))$ is an asymptotic interpolating sequence, so we may use Lemma 3.3 to obtain functions $\tilde{k}_m \in \mathscr{B}$ satisfying the conditions of that lemma for $(\phi_m^{-1}(z_0))$. Consider the function $h_j = (1+\tilde{k}_j)/2$. Then $h_j \circ \phi_j^{-1} \to 1$ and $h_j \circ \phi_j^{-1} \circ \phi_m^{-1} \underset{m\to\infty}{\to} 0$ uniformly on compacta.

Let

$$D_1 = \{d_n := p_n(g_n \circ \phi_n)^2 : n \in \mathbb{N}\}$$

and

$$D_2 = \{\tilde{d}_j := q_j(h_j \circ \phi_j^{-1}) : j \in \mathbb{N}\}.$$

Since $g_n \circ \phi_n \to 1$ and $h_j \circ \phi_j^{-1} \to 1$ uniformly on compacta, both $D_1$ and $D_2$ are dense in $\mathscr{B}$ (Here we have used the fact that $\mathscr{B}$, as a convex set in a topological vector space, does not contain any isolated point). Finally, let

$$d_{n,m} := (1 - k_n \circ \phi_n \circ \phi_m)^2/4.$$

Then $d_{n,m} \to 1$ as $m \to \infty$. Note that $d_n = p_n(1 + k_n \circ \phi_n)^2/4$. Since

$$\frac{1}{4}\left[|1 + k_n \circ \phi_n|^2 + |1 - k_n \circ \phi_n|^2\right] \leq 1,$$

we conclude that $d_n + \tilde{d}_j d_{n,m} \circ \phi_m^{-1} \in \mathscr{B}$ for every $n, m, j \in \mathbb{N}$.

Moreover $C_{\phi_m}(d_n) = (p_n \circ \phi_m)(g_n \circ \phi_n \circ \phi_m)^2 \to 0$ as $m \to \infty$ since $g_n \circ \phi_n \circ \phi_m \to 0$ as $m \to \infty$. Similarly, $C_{\phi_m^{-1}}(\tilde{d}_j) = \left(q_j \circ \phi_m^{-1}\right)\left(h_j \circ \phi_j^{-1} \circ \phi_m^{-1}\right) \to 0$ as $m \to \infty$.

(b) $\Longrightarrow$ (c): By Theorem 2.7 there exists a $G_\delta$-set of $\mathscr{B}$-biuniversal functions for the sequence $(\phi_{n_k})$ that is dense in $\mathscr{B}$. Take one such $\mathscr{B}$-universal function, $F$. By Lemma 3.1, there exists a subsequence $(\phi_{n_{k_\ell}})$ so that $(\phi_{n_{k_\ell}}(z_0))_\ell$ is an asymptotic interpolating sequence of type one. By assumption, Theorem 2.7 applies. Thus we get a $\mathscr{B}$-biuniversal function for the sequence $(\phi_{n_{k_\ell}})$.

(c) $\Longrightarrow$ (a): Let $(\phi_{n_j}(z_0))$ be an asymptotic interpolating sequence of type one and $f$ a $\mathscr{B}$-biuniversal function of norm one such that the set $\{f \circ \phi_{n_j}^{-1} : j \in \mathbb{N}\}$ is dense in $\mathscr{B}$. So, for some subsequence, $(\phi_{n_{j_k}}^{-1})$, we see that $f \circ \phi_{n_{j_k}}^{-1} \to 1$ uniformly on compacta. By Lemma 2.3, $(\phi_{n_{j_k}}^{-1}(z_0))$ contains an asymptotic interpolating sequence of type one.

Now we establish the existence of universal functions for invertible composition operators $C_{\phi_n}$ for which $(\phi_n(z_0))$ contains an asymptotic interpolating sequence of type one for some $z_0 \in \Omega$, but no information is available about $(\phi_n^{-1})$.

**Theorem 3.5** *Suppose that $\Omega \subseteq \mathbb{C}^N$ is a domain for which $H^\infty(\Omega)$ is nontrivial. Let $(\phi_n)$ be a sequence of automorphisms of $\Omega$. Then there exists a $\mathscr{B}$-universal function for $(\phi_n)$ if and only if for some $z_0 \in \Omega$ (and hence for all $z \in \Omega$) the sequence $(\phi_n(z_0))$ contains an asymptotic interpolating sequence of type one.*

*Proof* If there exists a $\mathscr{B}$-universal function for $(\phi_n)$, then Lemma 3.1 implies that $(\phi_n(z_0))$ contains an asymptotic interpolating sequence of type one.

Now suppose that $(\phi_n(z_0))$ contains an asymptotic interpolating sequence of type one (which we denote with the index $n$ again). We claim that there is a sequence $f_n \in \mathscr{B}$ such that $f_n \to 0$ and $f_n \circ \phi_n \to 1$ uniformly on compacta. We will use $f_n$ to build our universal functions. To verify our claim, use Lemma 2.5 to choose $f \in \mathscr{B}$ so that $f(\phi_n(z_0)) \to 1$ and

$f(z_0) = 0$. Then, by definition of the pseudohyperbolic distance, $\rho_\Omega(z_0, \phi_n(z_0)) \to 1$. By Lemma 2.6, there exist $h_n \in \mathscr{B}$ with $h_n(z_0) \to -1$ and $h_n(\phi_n(z_0)) \to 1$. By the maximum principle, $h_n \to -1$ and $h_n \circ \phi_n \to 1$ uniformly on compacta in $\Omega$. Let $f_n = \left(\frac{1+h_n}{2}\right)^2$. Then $f_n \in \mathscr{B}$, $f_n \to 0$ and $f_n \circ \phi_n \to 1$, as claimed.

Let $\{p_m : m \in \mathbb{N}\}$ be dense in $\mathscr{B}$ in the compact open topology. We take our motivation from Theorem 1.3: we construct functions $g_{n,m}$ such that $g_{n,m} \to 0$ as $n \to \infty$ and $g_{n,m} \circ \phi_n \to p_m$. To do this, define the functions $g_{n,m}$ on $\Omega$ by

$$g_{n,m} = \left(p_m \circ \phi_n^{-1}\right) f_n. \tag{3.1}$$

Then, because $p_m$ is a bounded function, $g_{n,m} \to 0$ as $n \to \infty$, and, for every $m$, we have $g_{n,m} \circ \phi_n \to p_m$ as $n \to \infty$.

By construction, the set

$$D_1 = \left\{ F_{n,j} := \left(\frac{1-h_n}{2}\right)^2 p_j : n, j \in \mathbb{N} \right\}$$

is dense in $\mathscr{B}$. Moreover,

$$|g_{n,m}| + |F_{n,j}| \leq |f_n| + |F_{n,j}| \leq |(1+h_n)^2/4| + |(1-h_n)^2/4| \leq 1. \tag{3.2}$$

Therefore, $g_{n,m} + F_{n,j} \in \mathscr{B}$.

We now verify that $\{C_{\phi_n} : n \in \mathbb{N}\}$ satisfies Birkhoff's Universality Criterion for Baire spaces. So let $U$ and $V$ be open sets in $\mathscr{B}$. We must show that there exists $n$ such that $C_{\phi_n}(U) \cap V \neq \emptyset$. Let $u \in U$. Choose $\varepsilon > 0$ so that $B_d(u, \varepsilon) \cap \mathscr{B} \subseteq U$. Since $V$ is open, we may choose $p_m \in V$. Now $D_1$ is dense in $\mathscr{B}$, so this together with the definition implies that there exists $j_0$ and $n_0$ such that $F_{n,j_0} \in B_d(u, \varepsilon/2) \cap \mathscr{B} \subseteq U$ for $n > n_0$. Now consider the functions

$$G_{n,m,j_0} = g_{n,m} + F_{n,j_0}.$$

By (3.2), $G_{n,m,j_0} \in \mathscr{B}$ for each $n$ and $m$. But $g_{n,m} \to 0$ as $n \to \infty$, so there exists $n_1$ such that $||g_{n,m}||_d < \varepsilon/2$ for $n \geq n_1$. Hence, for $n \geq \max\{n_0, n_1\}$, we have

$$||G_{n,m,j_0} - u||_d \leq ||g_{n,m}||_d + ||F_{n,j_0} - u||_d \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So we may choose $n$ so large that $G_{n,m,j_0} \in U$. On the other hand, since $h_n \circ \phi_n \to 1$,

$$F_{n,j_0} \circ \phi_n = \left(\frac{1-h_n\circ\phi_n}{2}\right)^2 (p_{j_0} \circ \phi_n) \to 0 \text{ as } n \to \infty$$

so

$$G_{n,m,j_0} \circ \phi_n = g_{n,m} \circ \phi_n + F_{n,j_0} \circ \phi_n \to p_m \in V,$$

as $n \to \infty$. Therefore, for $n$ large, we must have

$$G_{n,m,j_0} \circ \phi_n \in V.$$

Consequently, $C_{\phi_n}(U) \cap V \neq \emptyset$, concluding the proof.

**Corollary 3.6** *Let $(\phi_n)$ be a sequence of automorphisms of a domain $\Omega \subseteq \mathbb{C}^N$. Then the existence of a $\mathscr{B}$-universal function for $(\phi_n)$ implies that for some subsequence, $(\phi_{n_k})$, of $(\phi_n)$ the composition operators $C_{\phi_{n_k}}$ satisfy the hypothesis of Theorem 2.7* (b).

*Proof* Let $f$ be a $\mathscr{B}$-universal function for $(\phi_n)$ and fix $z_0 \in \Omega$. Then, for some subsequence, $(\phi_{n_k}(z_0))$ is an asymptotic interpolating sequence of type one. By Theorem 3.5, there exists a $\mathscr{B}$-universal function for the sequence $(\phi_{n_k})$. This means that whenever $U$ and $V$ are open sets in $\mathscr{B}$, there exists $k$ such that $C_{\phi_{n_k}}(U) \cap V \neq \emptyset$. This implies that $U \cap C^{-1}_{\phi_{n_k}}(V) \neq \emptyset$. Hence, by Birkhoff's transitivity criterion, there exists a $\mathscr{B}$-universal function for $\left(\phi^{-1}_{n_k}\right)$. Thus, for some subsequence, $(\phi^{-1}_{n_{k_\ell}}(z_0))$ is an asymptotic interpolating sequence of type one. Of course, as a subsequence of $(\phi_{n_k}(z_0))$, the sequence $(\phi_{n_{k_\ell}}(z_0))$ is an asymptotic interpolating sequence of type one, too. Hence, by Theorem 3.4, (a) $\Longrightarrow$ (b), we see that $(C_{\phi_{n_k}})$ satisfies the hypothesis of Theorem 2.7 (b)

Bernal-Gonzalez and Montes-Rodriguez ([3,21]) showed that for a planar domain $\Omega$ not conformally equivalent to $\mathbb{C} \setminus \{0\}$ (see also [2, p. 24]) there exists a function $f \in H(\Omega)$ for which $\{f \circ \phi_n : n \in \mathbb{N}\}$ is dense in $H(\Omega)$ if and only if the sequence of automorphisms $(\phi_n)$ is a runaway sequence. The corollary below complements and, under certain conditions, extends this result to domains in $\mathbb{C}^N$.

**Corollary 3.7** *Let $\Omega$ be a domain in $\mathbb{C}^N$ such that $H^\infty(\Omega)$ is locally uniformly dense in $H(\Omega)$. Suppose that $(\phi_n)$ is a sequence of automorphisms of $\Omega$ such that for some $z_0 \in \Omega$ the sequence $(\phi_n(z_0))$ contains an asymptotic interpolating subsequence of type one. Then there exists a function $f \in H(\Omega)$ for which the set $\{f \circ \phi_n : n \in \mathbb{N}\}$ is locally uniformly dense in $H(\Omega)$.*

*Proof* Suppose that $H^\infty(\Omega)$ is dense in $H(\Omega)$. Then we can choose a countable dense set $\{p_m : m \in \mathbb{N}\}$ of bounded functions in $H(\Omega)$. Without loss of generality we may assume that $(\phi_n(z_0))$ is an asymptotic interpolating sequence of type one. Let $k_n$ be the functions constructed in Lemma 3.3 and let $g_n = (1+k_n)/2$. Then $g_n \circ \phi_n \to 1$ and $g_n \circ \phi_n \circ \phi_m \to 0$ as $m \to \infty$. Hence $D_1 = \{p_n(g_n \circ \phi_n) : n \in \mathbb{N}\}$ is dense in $H(\Omega)$ and $C_{\phi_m}$ converges to 0 on $D_1$. As in the proof of Theorem 3.5, (see (3.1)), we consider the functions $t_{n,m} = (p_m \circ \phi_n^{-1}) f_n$. They satisfy $t_{n,m} \to 0$ as $n \to \infty$ and $t_{n,m} \circ \phi_n \to p_m$ as $n \to \infty$. Hence, by Theorem 1.3, this yields a $G_\delta$-set of $H(\Omega)$-universal functions.

Examples of sets in $\mathbb{C}$ for which $H^\infty(\Omega)$ is locally uniformly dense in $H(\Omega)$ include planar domains for which all components of the complement have nonempty interior. Domains for which $H^\infty(\Omega)$ is not dense include sets $\Omega \subseteq \mathbb{C}$ for which the complement has components that are Painlevé null sets, for example, $\mathbb{D} \setminus \{0\}$.

## 4 Constructing universal functions

Thus far, we have shown how the existence of universal functions can be derived from the universality theorems. In this section, we show how $\mathscr{B}$-universal functions can be constructed.

**Lemma 4.1** *Assume that $\Omega \subseteq \mathbb{C}^N$ is a domain for which $H^\infty(\Omega)$ is non-trivial. Let $(w_n)$ be an asymptotic interpolating sequence of type one in $\Omega$ and let $\varepsilon_{jk} > 0$ for each $j, k \in \mathbb{N}$. Then there exists a subsequence $(z_j)$ of $(w_n)$ and functions $f_j$ and $g_j$ in $\mathscr{B}$ satisfying:*

$$f_j(z_j) \to 1 \text{ and } g_j(z_j) \to 0, \tag{4.1}$$

$$|f_j| + |g_j| \le 1, \tag{4.2}$$

$$f_j(z_k) = 0 \text{ for } k < j, \tag{4.3}$$

$$|f_j(z_k)| \le (\varepsilon_{jk})^2 \text{ for } k > j, \tag{4.4}$$

$$|1 - g_j(z_k)| < 2\varepsilon_{jk} \text{ for } k > j. \tag{4.5}$$

*Proof* The proof will be by induction. Let $(\varepsilon_j)$ be a sequence of positive numbers satisfying $\varepsilon_j \to 0$. Let $n = 1$. Since $(w_j)$ is an asymptotic interpolating sequence of type one, there exists a nonconstant function $p \in \mathscr{B}$ such that $p(w_j) \to -1$. Choose $z_1 \in \{w_j : j \in \mathbb{N}\}$ so that $|p(z_1)| > 1 - 2\varepsilon_1$. By composing on the left with a Möbius transformation, $\psi_1$, of $\mathbb{D}$ onto $\mathbb{D}$ satisfying $\psi_1(-1) = -1$ and $\psi_1\big(p(z_1)\big) = |p(z_1)|$, we see that at $z_1$ the value of $p_1 := \psi_1 \circ p$ is positive. Let $f_1 = \left(\frac{1+p_1}{2}\right)^2$ and $g_1 = \left(\frac{1-p_1}{2}\right)^2$. Then $|f_1| + |g_1| \leq 1$, $|f_1(z_1)| > (1-\varepsilon_1)^2$, $|g_1(z_1)| \leq \varepsilon_1^2$, and $f_1(w_j) \to 0$.

Now assume that for $j = 1, \dots, n-1$ the functions $p_j$, $f_j$ and $g_j$ and the points $z_j \in \{w_m : m \in \mathbb{N}\}$ have been chosen so that $f_j = B_j \left(\frac{1+p_j}{2}\right)^2$ and $g_j = \left(\frac{1-p_j}{2}\right)^2$ for some $B_j \in \mathscr{B}$, where $p_j = \psi_j \circ p$ for some holomorphic self-map $\psi_j$ of $\mathbb{D}$ fixing the point $-1$ and sending $p(z_j)$ to $|p(z_j)|$.

For $a \in \mathbb{D}$, let $\psi_a(\xi) = \frac{a-\xi}{1-\overline{a}\xi}$ and for $j = 1, \dots, n-1$, let $b_j := \psi_{p(z_j)} \circ p$ and $B_n := b_1 \cdots b_{n-1}$. Then $B_n(z_j) = 0$ for $j = 1, \dots, n-1$. Since $p(w_m) \to -1$ as $m \to \infty$, we may choose $z_n \in \{w_m : m \in \mathbb{N}\} \setminus \{z_1, \dots, z_{n-1}\}$ such that

$$|B_n(z_n)| \sim 1 \text{ and } |p(z_n)| \sim 1, \tag{4.6}$$

and

$$|p_j(z_n) + 1| < 2\varepsilon_{jn} \text{ for } j = 1, \dots, n-1. \tag{4.7}$$

Composing on the left with a Möbius transformation $\psi_n$ satisfying $\psi_n(-1) = -1$ and $\psi_n(p(z_n)) = |p(z_n)|$, we obtain a function $p_n := \psi_n \circ p$ whose value at $z_n$ is positive and close to one.

Let $f_n = B_n\left(\frac{1+p_n}{2}\right)^2$ and $g_n = \left(\frac{1-p_n}{2}\right)^2$. Then $f_n(z_j) = 0$ for $j = 1, \dots, n-1$, $|f_n| + |g_n| \leq 1$, $f_n(z_n) \sim 1$, $g_n(z_n) \sim 0$, $f_n(w_m) \to 0$ as $m \to \infty$. Moreover

$$|f_j(z_n)| \leq \left|\frac{1+p_j(z_n)}{2}\right|^2 \leq (\varepsilon_{jn})^2$$

and

$$|1 - g_j(z_n)| = \left|1 - \left(\frac{1-p_j(z_n)}{2}\right)^2\right| = \left|1 + \left(\frac{1-p_j(z_n)}{2}\right)\right| \left|1 - \left(\frac{1-p_j(z_n)}{2}\right)\right|$$

$$\leq 2\left|\frac{1+p_j(z_n)}{2}\right| \leq 2\varepsilon_{jn}.$$

whenever $1 \leq j \leq n-1$.

We remark that if, in the lemma above, the sequence $(w_n)$ has the form $w_n = \phi_n(z_0)$ for some self-map $\phi_n$ of $\Omega$, then for some subsequence $(\phi_{n_j})$, the functions $f_n \circ \phi_{n_j}$ tend to 0 locally uniformly in $\Omega$ as $j \to \infty$. Indeed, since $z_j$ is a subsequence of $w_n$, we have $z_j = \phi_{n_j}(z_0)$ and so

$$p_n(\phi_{n_j}(z_0)) = p_n(z_j) = \psi_n(p(z_j)) \to \psi_n(-1) = -1.$$

Since $||p_n||_\infty \leq 1$, we know that $p_n \circ \phi_{n_j} \to -1$ uniformly on compacta as $j \to \infty$. Hence

$$f_n \circ \phi_{n_j} = (B_n \circ \phi_{n_j}) \left(\frac{1 + p_n \circ \phi_{n_j}}{2}\right)^2$$

tends to 0 uniformly on compacta as $j \to \infty$.

The following lemma is a version of a result in [1].

**Lemma 4.2** *Let $a_j, b_j$ be complex numbers such that $|a_j| + |b_j| \le 1$. Then for every $N$*

$$|a_1| + |a_2 b_1| + |a_3 b_1 b_2| + \cdots + |a_N b_1 b_2 \dots b_{N-1}| \le 1.$$

The following result will be crucial to our construction of universal functions.

**Theorem 4.3** *Assume that $\Omega \subseteq \mathbb{C}^N$ is a domain for which $H^\infty(\Omega)$ is non-trivial. Let $(\phi_n)$ be a sequence of self-mappings of $\Omega$. Suppose that for some $z_0 \in \Omega$ the sequence $(\phi_n(z_0))$ contains an asymptotic interpolating sequence of type one for $H^\infty(\Omega)$. Then there exists a subsequence $(\phi_{n_j})$ of $(\phi_n)$ such that for any sequence $(h_j)$ of functions in $\mathscr{B}$ there exists a function $F \in \mathscr{B}$ such that $|F \circ \phi_{n_j} - h_j \circ \phi_{n_j}| \to 0$ uniformly on compacta in $\Omega$.*

*Proof* Let $(w_m)$ be a subsequence of $(\phi_n(z_0))$ that is an asymptotic interpolating sequence of type one for $H^\infty(\Omega)$. Use Lemma 4.1 to choose functions $f_j$ and $g_j$ and a subsequence $(z_j)$ of $(w_m)$ satisfying (4.1)–(4.5) with $\epsilon_{jk} = 1/2^{j+k+1}$. Let $g_0 := 1$. By Lemma 4.2, we have that $\sum_{n=1}^\infty |f_n g_1 \cdots g_{n-1}| \le 1$. We show that the function

$$F = \sum_{n=1}^{\infty} h_n (f_n g_1 \cdots g_{n-1})$$

is the desired function. Applying Montel's theorem and Vitali's theorem, we conclude that $F \in H^\infty(\Omega)$ and $||F||_\infty \le 1$. Note that by the choice of the $z_j$ we have $z_j = \phi_{n_j}(z_0)$ for some $n_j$.

**Claim** $\big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}$ tends to 1 uniformly on compacta.

To show this, note that

$$\big(f_j g_1 \cdots g_{j-1}\big)(\phi_{n_j}(z_0)) = \big(f_j g_1 \cdots g_{j-1}\big)(z_j).$$

We will apply Lemma 4.1. Applying the inequality $|1 - \prod_k a_k| \le \sum_k |a_k - 1|$ for $|a_k| \le 1$, we obtain

$$\begin{aligned} \big|\big(f_j g_1 \cdots g_{j-1}\big)(z_j) - 1\big| &= |f_j(z_j) \prod_{k=1}^{j-1} g_k(z_j) - 1| \le \\ &\le |f_j(z_j) - 1| + \sum_{k=1}^{j-1} |g_k(z_j) - 1| \underset{(4.1),(4.5)}{\le} \varepsilon_j + 2\sum_{k=1}^{j-1} \varepsilon_{kj} = \\ &= \varepsilon_j + 2\sum_{k=1}^{j-1} 2^{-k-j-1} \le \varepsilon_j + 2^{-j} \to 0 \text{ as } j \to \infty. \end{aligned}$$

But

$$|\big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}| \le 1.$$

By a normal families argument and the maximum principle, we conclude that the sequence $\big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j} \to 1$ uniformly on compacta. This completes the proof of the claim.

Now we evaluate $F$ at $z_j$ to obtain

$$\begin{aligned} F(z_j) = \textstyle\sum_{n:n<j} h_n(z_j)(f_n g_1 \cdots g_{n-1})(z_j) + h_j(z_j)(f_j g_1 \cdots g_{j-1})(z_j) + \\ + \textstyle\sum_{n:n>j} h_n(z_j)(f_n g_1 \cdots g_{n-1})(z_j). \end{aligned}$$

By (4.3) the third summand is zero and by (4.4) the first summand is majorized by

$$\sum_{n:n<j} 2^{-n-j-1} < 2^{-j}.$$

Hence

$$\lim_{j\to\infty} |F(z_j) - h_j(z_j)\,(f_j g_1 \cdots g_{j-1})(z_j)| = 0. \tag{4.8}$$

Note that if we write $w = \phi_{n_j}(z)$ we get

$$\begin{aligned}\big|F \circ \phi_{n_j}(z) - (h_j \circ \phi_{n_j})(z)[(f_j g_1 \cdots g_{j-1}) \circ \phi_{n_j}(z)] + (f_j g_1 \cdots g_{j-1}) \circ \phi_{n_j}(z)\big| \\ \le \textstyle\sum_{n\neq j} |(h_n f_n g_1 \cdots g_{n-1})(w)| + |(f_j g_1 \cdots g_{j-1})(w)| \\ \le \textstyle\sum_{n=1}^{\infty} |(f_n g_1 \cdots g_{n-1})(w)| \le 1.\end{aligned}$$

In other words,

$$\big|F \circ \phi_{n_j} - (h_j \circ \phi_{n_j})\big[\big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}\big] + \big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}\big| \le 1. \tag{4.9}$$

Evaluating at $z_0$, recalling that $z_j = \phi_{n_j}(z_0)$ in (4.8), and using the fact that $(f_j g_1 \cdots g_{j-1})(\phi_{n_j}(z_0))$ tends to 1, we see that

$$\Big(F \circ \phi_{n_j} - (h_j \circ \phi_{n_j})\big[\big(f_j g_1 \cdots g_{j-1}\big)\circ\phi_{n_j}\big] + \big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}\Big)(z_0) \to 1.$$

Again by the maximum principle, (4.9) implies that

$$F \circ \phi_{n_j} - (h_j \circ \phi_{n_j})\big[\big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}\big] + \big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j} \to 1$$

uniformly on compacta. Since $(f_j g_1 \cdots g_{j-1}) \circ \phi_{n_j}$ tends to 1, we deduce that

$$F \circ \phi_{n_j} - (h_j \circ \phi_{n_j})\big[\big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}\big] \to 0$$

uniformly on compacta. Thus

$$\begin{aligned}|F \circ \phi_{n_j} - h_j \circ \phi_{n_j}| &\le \big|F \circ \phi_{n_j} - (h_j \circ \phi_{n_j})\big[\big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}\big]\big| + \\ &+ \big|h_j \circ \phi_{n_j} - (h_j \circ \phi_{n_j})\big[\big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}\big]\big| \le \\ &\le \big|F \circ \phi_{n_j} - (h_j \circ \phi_{n_j})\big[\big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}\big]\big| + \big|1 - \big(f_j g_1 \cdots g_{j-1}\big) \circ \phi_{n_j}\big| \to 0\end{aligned}$$

uniformly on compacta. This proves that, with the compact open topology on $H(\Omega)$, the accumulation points (in $H(\Omega)$) of the sets $\{F \circ \phi_{n_j} : j \in \mathbb{N}\}$ and $\{h_j \circ \phi_{n_j} : j \in \mathbb{N}\}$ are the same.

We point out that the subsequence $(\phi_{n_j})$ is independent of the approximating sequence $(h_m)$. This subsequence, which came from Lemma 4.1, was chosen so that $\phi_{n_j}(z_0)$ tended very quickly to the boundary.

We are now ready to exhibit, in the most general setting, an explicit formula for a $\mathscr{B}$-universal function associated with the composition operators $C_{\phi_n}$ on $H^\infty(\Omega)$. The following proofs stand in contrast to the proofs of Theorem 3.5 in that Theorem 3.5 relied solely on our universality criteria (Theorem 2.7), while the proof of Theorem 4.4, below, is constructive.

**Theorem 4.4** *Let $\Omega \subseteq \mathbb{C}^N$ be a domain for which $H^\infty(\Omega)$ is non-trivial. Suppose that $(\phi_n)$ is a sequence of automorphisms such that $(\phi_n(z_0))$ contains an asymptotic interpolating sequence of type one for $H^\infty(\Omega)$. Then there exists a $\mathscr{B}$-universal function $F$; that is, there is a function $F \in \mathscr{B}$ for which the set $\{F \circ \phi_n : n \in \mathbb{N}\}$ is locally uniformly dense in $\mathscr{B}$.*

*Proof* Since $H^\infty(\Omega)$ is separable, its unit ball $\mathscr{B}$ is separable, too. Hence there exists a countable dense subset $\{p_j : j \in \mathbb{N}\}$ in $\mathscr{B}$. Let $\phi_{n_j}$ be the subsequence above. Define $h_j = p_j \circ \phi_{n_j}^{-1}$. Then $h_j \in \mathscr{B}$ and $p_j = h_j \circ \phi_j$. By Theorem 4.3 there exists a function $F \in \mathscr{B}$ such that $|F \circ \phi_{n_j} - h_j \circ \phi_{n_j}| = |F \circ \phi_{n_j} - p_j| \to 0$ locally uniformly in $\Omega$. Then $F$ is the desired universal function.

## 5 Asymptotic interpolating sequences and runaway automorphisms

If we assume our domain is bounded, we can recover a standard approach in this area. We let $\hat{\mathbb{C}}$ denote the extended complex plane.

**Corollary 5.1** *Assume that $\Omega \subset \mathbb{C}$ is a bounded domain such that:*

1. *the boundary of $\Omega$ does not contain any component that is a singleton and*
2. *there exists a runaway sequence of automorphisms $(\phi_n)$ of $\Omega$.*

*Then there exists a function $f$ in the unit ball $\mathscr{B}$ of $H^\infty(\Omega)$ such that $\{f \circ \phi_n : n \in \mathbb{N}\}$ is dense in $\mathscr{B}$ in the compact open topology.*

*Proof* Suppose that for every compact set $K \subseteq \Omega$ there exists an integer $n = n(K)$ such that $\phi_m(K) \cap K = \emptyset$ for all $m \geq n(K)$. In particular, for every $z \in \Omega$, the sequence $(\phi_n(z))$ converges to the boundary of $\Omega$, denoted $\partial\Omega$. Fix a point $z_0 \in \Omega$. Since $\Omega$ is bounded, there exists $\xi \in \partial\Omega$ and a subsequence $w_k = \phi_{n_k}(z_0)$ such that $w_k \to \xi$. Let $E_\xi$ be the connected component of $\hat{\mathbb{C}} \setminus \Omega$ that contains $\xi$. Then $E_\xi$ is closed and $\xi$ is a boundary point of $E_\xi$. By a topological argument (see [22, p. 78, Theorem 3.2]), the set $U := \hat{\mathbb{C}} \setminus E_\xi$ is a connected set in $\hat{\mathbb{C}}$ and $\Omega \subseteq U$. Since $E_\xi$ is connected, $U$ is a simply connected domain in $\hat{\mathbb{C}}$. Let $R$ be the Riemann map of $U$ onto the unit disk $\mathbb{D}$ with $R(\infty) = 0$. Now, there exists a subsequence $(w_{k_j})$ of $(w_k)$ such that $R(w_{k_j})$ converges to the boundary of $\mathbb{D}$, say $\lim_j R(w_{k_j}) = 1$. Let $p(z) = (1+z)/2$ be the peak function for $A(\mathbb{D})$ associated with the point 1 and let $f = p \circ R$. Then $f \in H^\infty(\Omega)$ and has norm one. Moreover, $\lim f(w_{k_j}) = 1$ and $|f(w_{k_j})| < 1$. Hence, by Lemma 2.3, $(w_{k_j})$ admits an asymptotic interpolating sequence of type one and, therefore, so does $(\phi_n(z_0))$. Consequently, Theorem 4.4 implies the result.

Example 2 in the next section shows that there exist infinitely connected unbounded domains $\Omega \subseteq \mathbb{C}$ such that the iterates of an automorphism converge to a boundary point that is a non-isolated component of the boundary of $\Omega$.

We will say that a boundary point $\xi$ of a bounded domain $\Omega \subseteq \mathbb{C}^N$ is an $H^\infty(\Omega)$-peak point if there exists a function $f$ of norm one, called a peak function, such that $\lim_{\substack{z\to\xi \\ z\in\Omega}} f(z) = 1$ and $\limsup_{\substack{z\to\eta \\ z\in\Omega}} |f(z)| < 1$ for every boundary point $\eta \neq \xi$.

**Proposition 5.2** *Let $\Omega \subseteq \mathbb{C}^N$ be a bounded domain such that every boundary point is an $H^\infty(\Omega)$-peak point. Then the existence of a runaway sequence of automorphisms for $\Omega$ implies the existence of $\mathscr{B}$-universal functions.*

*Proof* If $(\phi_n)$ is a runaway sequence, then given $z_0 \in \Omega$, there exists a subsequence $\left(\phi_{n_k}\right)$ of $(\phi_n)$ such that $\left(\phi_{n_k}(z_0)\right)$ converges to a boundary point $\xi$. By our hypothesis, $\xi$ is a peak point. The associated peak function $g$ satisfies $\|g\|_\infty < 1$ and $g\left(\phi_{n_k}(z_0)\right) \to 1$. By Lemma 2.3, $\left(\phi_{n_k}(z_0)\right)$ has an asymptotic interpolating subsequence and so Theorem 4.4 implies the result.

**Theorem 5.3** *Let $\Omega$ be a bounded domain in $\mathbb{C}$. Suppose that $(\phi_n)$ is a sequence of automorphisms of $\Omega$ and let $z_0 \in \Omega$. Then the following are equivalent.*

(a) *The sequence* $(\phi_n(z_0))$ *contains an asymptotic interpolating subsequence of type one.*
(b) *There is a* $\mathscr{B}$*-universal function for the sequence* $(\phi_n)$.
*Each of these conditions implies*
(c) *The sequence* $(\phi_n)$ *is runaway.*

*If, in addition,* $\Omega$ *satisfies condition* (1) *of Corollary* 5.1*, then* (a), (b) *and* (c) *are equivalent.*

*Proof* The equivalence of (a) and (b) follows from Theorem 4.4 and Lemma 3.1.

We prove that (a) implies (c): Without loss of generality, let us assume that $(\phi_n(z_0))$ is an asymptotic interpolating sequence of type one. Suppose that $(\phi_n)$ is not runaway. Then there exists a compact set $K \subseteq \Omega$ such that for all $n$ we have $\phi_n(K) \cap K \neq \emptyset$. Hence there is a sequence $(\xi_n)$ in $K$ such that $\phi_n(\xi_n) \in K$ for all $n$. Let $n_j$ be chosen so that $\lim \phi_{n_j}(\xi_{n_j}) =: \eta_0$ exists. Let $w_n := \phi_n(z_0)$. Since $(w_n)$ is assumed to be an asymptotic interpolating sequence of type one, there exists a function $H$, of norm one, such that $H(w_n) \to 1$ and $|H(w_n)| < 1$ for all $n$. Consider the sequence of functions $(H \circ \phi_n)$. Since this sequence is bounded by 1 and converges to a value of modulus 1 at $z_0 \in \Omega$, we know that the sequence converges to the value 1 on compact subsets of the domain. Thus, we conclude that for each $\epsilon > 0$, there exists $n(K)$ such that $\sup_{z\in K} |H \circ \phi_n(z) - 1| < \epsilon$ for $n > n(K)$. Therefore $(H \circ \phi_{n_j})(\xi_{n_j}) \to 1$. In particular, $H(\eta_0)$ has modulus 1. Thus, applying the maximum modulus theorem, we conclude that $H$ is constant. This contradiction implies that the sequence $(\phi_n(z_0))$ is runaway; hence we get (c). Now, if additionally condition (1) of Corollary 5.1 holds, then the same corollary implies that (b) holds, completing the proof of this theorem.

## 6 Examples

Let Aut($\Omega$) denote the group of automorphisms on the domain $\Omega$. We note that a finitely connected domain in $\mathbb{C}$ with at least two holes cannot have a sequence of runaway automorphisms (see [16]), because Aut($\Omega$) is compact in these cases. However, an infinitely connected set can have such a sequence. Kim and Krantz's recent article [18] contains many relevant examples and comments.

Combining information from several sources, we obtain the following for an infinitely connected domain $\Omega \subset \mathbb{C}$.

**Proposition 6.1** *For an infinitely connected domain* $\Omega \subset \mathbb{C}$*, the following are equivalent.*

1. *Aut(*$\Omega$*) is infinite.*
2. *Aut(*$\Omega$*) contains a runaway sequence.*
3. *Aut(*$\Omega$*) is not compact.*
4. *There exists a sequence* $(\phi_n)$ *in Aut(*$\Omega$*) such that for some (every)* $z_0 \in \Omega$ *the sequence* $(\phi_n(z_0))$ *tends to* $\partial\Omega$.
5. *For all* $z_0 \in \Omega$ *there exists* $\phi_n \in$ *Aut(*$\Omega$*) and* $\xi \in \partial\Omega$ *such that* $\phi_n(z_0) \to \xi$.

*Proof* The equivalence of (1) and (2) appears in ([21, p. 194]). The equivalence of (3) and (4) can be found in [18]. It is clear that (3) implies (1), (2) implies (5), and (5) implies (4).

The following example of an infinitely connected domain with a runaway sequence of automorphisms, appeared in [18].

*Example 1* [18, Example 2.1] Let $\mathbb{D}$ be the unit disk in the plane and $K = \{z \in \mathbb{C} : |z| \leq 1/10\}$. The automorphism $\phi$ is defined by

$$\phi(z) = \frac{z - 1/2}{1 - (1/2)z}.$$

Let $\phi^{[j]}$ denote the $j$-fold iterate, ($j \in \mathbb{Z}$). The domain $\Omega$ is given by

$$\Omega := \mathbb{D} \setminus \bigcup_{j=-\infty}^{\infty} \phi^{[j]}(K).$$

Since each $\phi^{[j]}$ is an automorphism of $\Omega$, it is clear that there are infinitely many automorphisms of $\Omega$. Further, it is noted in [18] that

$$\phi^{[j]}(z) = \frac{z - \frac{3^j-1}{3^j+1}}{1 - \frac{3^j-1}{3^j+1}z}.$$

Therefore, for any point $z_0 \in \Omega$, $\lim_{j\to\infty} \phi^{[j]}(z_0) = -1$ and so $\phi^{[j]} \to -1$ uniformly on compacta. Thus, $(\phi^{[j]})$ is an example of a runaway sequence of automorphisms on an infinitely connected domain. By Corollary 5.1, $\Omega$ therefore admits $\mathscr{B}$-universal functions.

*Example 2* Let $P$ be the set of poles of the iterates $\phi^{[j]}$ above. Now let $\Omega$ be defined as

$$\Omega = \mathbb{C} \setminus \left[ \bigcup_{j=-\infty}^{\infty} \phi^{[j]}(K) \cup P \right].$$

Again, $\phi^{[j]} \in \text{Aut}(\Omega)$. Here the (positive) iterates converge to the boundary point $-1$, which is itself a component for $\partial\Omega$. Using the peak point criterion of Melnikov-Gamelin-Garnett [7], it can be shown that $-1$ is a peak point for $H^\infty(\Omega)$. Hence, by Lemma 2.3, the sequence $\left(\phi^{[n]}(z_0)\right)$ admits an asymptotic interpolating sequence of type one, and so, by Theorem 3.5, this domain $\Omega$ admits $\mathscr{B}$-universal functions.

By the results of this paper, if $\Omega \subseteq \mathbb{C}^N$ is a domain for which there exists a sequence $(\phi_n)$ of automorphisms and a non-constant function $f \in \mathscr{B}$ that is extremal on the orbit of $z_0$ under $(\phi_n)$, (i.e. if $|f(\phi_n(z_0))| \to 1$), then the sequence of composition operators $(C_{\phi_n})$ admits a $\mathscr{B}$-universal vector. Examples of such domains in $\mathbb{C}^N$ include the ball, the polydisk $\mathbb{D}^N$ (see [23, p. 167]) and the "complex ellipsoids"

$$E_{1,m} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$$

(see [18, p. 593, 597] and [17, p. 273]). Note that if $(\phi_n)$ is a sequence of automorphisms of the polydisk $\mathbb{D}^N$ such that $\phi_n(0)$ converges to a point $\alpha = (\alpha_1, \ldots, \alpha_N)$ of the topological boundary of $\mathbb{D}^N$ (say $\alpha_1 = 1$), then the function $p(z_1, \cdots, z_N) = (1 + z_1)/2$ is a function that converges to 1 on the orbit $(\phi_n(0))$; hence our theory can be applied. In particular the automorphisms $\phi_n(z, w) = \left(\frac{z+r_n}{1+r_n z}, w\right)$ admit $\mathscr{B}$-universal functions whenever $r_n \to 1$, although the orbits do not accumulate at points on the distinguished boundary $\mathbb{T} \times \mathbb{T}$. Let us also mention that in all of these three examples the analogue of Theorem 5.3 holds.

It is clear that if the automorphism group of a domain $\Omega$ in $\mathbb{C}^N$ is compact, then this domain does not carry $\mathscr{B}$ or $H(\Omega)$-universal functions. As an example in $\mathbb{C}^2$ we mention the complex ellipsoid

$$E_{2,2} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^4 < 1\},$$

(see [18, p. 595]).

We close the paper with a few remarks about composition operators for which the symbol is not an automorphism.

(1) In order that the iterates $\phi^{[n]}$ of a self-map $\phi$ of a domain $\Omega \subseteq \mathbb{C}^N$ admit a universal function $F$, $\phi$ must be injective. In fact, if it were the case that $\phi(z) = \phi(w)$, then $\phi^{[n]}(z) = \phi^{[n]}(w)$ for every $n$. Since $F$ is assumed to be universal, there would exist a subsequence $(\phi^{[n_j]})$ such that $F \circ \phi^{[n_j]}$ converges to the identity function uniformly on compacta and, therefore, $z = w$.

(2) A further necessary condition for a sequence $\phi_n : \mathbb{D} \to \mathbb{D}$ to admit a universal function is that $\limsup_n |\phi_n'(0)|/(1 - |\phi_n(0)|^2) = 1$. To see this, let $F$ be universal and say $F \circ \phi_n$ tends to the identity. Then, taking derivatives and evaluating at the origin we obtain

$$F'(\phi_n(0))\phi_n'(0) = \left[(1 - |\phi_n(0)|^2)F'(\phi_n(0))\right] \left[\frac{\phi_n'(0)}{1 - |\phi_n(0)|^2}\right] \to 1.$$

By Schwarz's lemma these two expressions are less than one in modulus. Hence $|\phi_n'(0)|/\left(1 - |\phi_n(0)|^2\right)$ must tend to 1.

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