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On multivariate dispersion orderings based on the standard construction

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Abstract

We study the relationship between the multivariate dispersive orders based on the standard construction. In particular those given by Shaked and Shanthikumar [1998. Two variability orders. Probab. Eng. Inform. Sci. 12, 1–23] and Fernández-Ponce and Suárez-Llorens [2003. A multivariate dispersion ordering based on quantiles more widely separated. J. Multivariate Anal. 85, 40–53]. In order to reach our objective we define a new weaker multivariate dispersive notion. Random vectors with a common copula and positive dependence properties are analyzed.

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1. Introduction

In the context of stochastic orders, several orders have been defined to compare two univariate random quantities in terms of their variability or dispersion. In particular, one of the most widely used in the literature is the dispersive order introduced by Lewis and Thompson (1981), see Shaked and Shanthikumar (1994) as an excellent handbook to study all its properties.

Based on the properties and characterizations of the univariate dispersive order several authors have proposed different multivariate extensions, namely Shaked and Shanthikumar (1998) and Fernández-Ponce and Suárez-Llorens (2003) propose multivariate dispersive orders based on the standard construction.

In this paper, we analyze the relationship between these multivariate dispersion notions. The organization is the following. First in Section 2 we present some well known characterizations of the univariate dispersion order and discuss how they have been used to define several multivariate dispersion orderings under the standard construction. Second in Section 3 we propose a new multivariate dispersion order which will be used later in Section 3 to relate the multivariate dispersion orders defined in Section 2. Third in Section 4 we show

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the relationship among the multivariate dispersion concepts introduced in the previous sections. In particular, some interesting results are obtained for random vectors with a common copula and positive dependence properties. Finally, in Section 5 we present how those relationships can be applied and used to simplify some known results in the literature.

In this paper for any random variable X and an event A, we let $\{X|A\}$ denote any random variable whose distribution is the conditional distribution of X given A. Expected values are assumed to exist whenever they are mentioned. By $=_{\mathrm{st}}$ we denote equality in law.

2. The univariate dispersion order: characterizations and generalizations

A formal definition of the univariate dispersive order is based on the notion of quantile. Given a random variable X with distribution function F, we define the *univariate quantile* as $Q_X(u) \equiv F_X^{-1}(u) = \inf\{x : F_X(x) \geqslant u\}$, for all real value $u \in (0,1)$ (for definition and characterizations of the dispersive order see Shaked and Shanthikumar, 1994, Section 2.B.2). Given two random variables X and Y, we say that X is less in the dispersive order than Y, denoted by $X \leqslant_{\mathrm{disp}} Y$, if

$$Q_X(v) - Q_X(u) \leqslant Q_Y(v) - Q_Y(u) \quad \text{for all } 0 < u \leqslant v < 1. \tag{1}$$

Clearly, this is a dispersive order notion because it requires the distance between any two quantiles of X to be less separated than the corresponding quantiles of Y.

An interesting characterization is given in terms of expansion functions. A real valued function ϕ is said to be an expansion function if $\phi(x') - \phi(x) \geqslant x' - x$ whenever $x' \geqslant x$. If ϕ is differentiable then ϕ is an expansion function if $\mathrm{d}\phi(x)/\mathrm{d}x \geqslant 1$ for all x. We have that $X \leqslant_{\mathrm{disp}} Y$ if, and only if,

$$Y =_{\mathrm{st}} \phi(X) \text{ for some expansion function } \phi. \tag{2}$$

Also, from the definition, it is easy to see that $X \leqslant_{\mathrm{disp}} Y$ if, and only if,

$$\phi(x) = Q_Y(F_X(x)) \text{ is an expansion function.} \tag{3}$$

In addition, if the distribution functions of X and Y are strictly increasing, then the function ϕ defined in (3), is the only one that satisfies (2). To finalize we present another characterization which we will use later. Given two random variables X and Y, with absolutely continuous distribution functions, and density functions f_X and f_Y, then $X \leqslant_{\mathrm{disp}} Y$ if, and only if,

$$f_X(Q_X(u)) \geqslant f_Y(Q_Y(u)) \quad \text{for all } u \in (0,1). \tag{4}$$

Based on these characterizations several authors have proposed different extensions to the multivariate case. Important contributions in this case have been made by Oja (1983) and Giovagnoli and Wynn (1995). Clearly, inspired in (2), those authors define multivariate dispersion orders through the existence of a multivariate function k which maps stochastically a random vector $\mathbf{X}$ to another one $\mathbf{Y}$, that is $\mathbf{Y} =_{\mathrm{st}} k(\mathbf{X})$. It is well known that there are different transformations that map a multivariate random vector to another one. For this reason Shaked and Shanthikumar (1998) and Fernández-Ponce and Suárez-Llorens (2003) consider a particular one based on the standard construction. First we need some definitions. From now on, we will assume that the multivariate distribution function is an absolutely continuous function.

Let $\mathbf{X}$ be a random vector and let $\mathbf{u} = (u_1, \ldots, u_n)$ in $[0,1]^n$. The standard construction for $\mathbf{X}$, denoted by

$$\hat{x}(\mathbf{u}) = (\hat{x}_1(u_1), \hat{x}_2(u_1, u_2), \ldots, \hat{x}_n(u_1, \ldots, u_n))$$

is defined as follows:

$$\hat{x}_1(u_1) = Q_{X_1}(u_1),$$
$$\hat{x}_i(u_1, \ldots, u_i) = Q_{\left\{X_i | \bigcap_{j=1}^{i-1} X_j = \hat{x}_j(u_j)\right\}}(u_i) \quad \text{for } i = 2, \ldots, n.$$

This known construction is widely used in simulation theory and plays the role of the quantile in the multivariate case. The following result, it is well known (see Li et al., 1996)

$$\hat{x}(\mathbf{U}) =_{st} \mathbf{X}, \tag{5}$$

where $\mathbf{U}$ is a random vector with n independent uniform distributed components on $[0,1]$.

By "inverting" the standard construction, we can express the independent uniform random variables U_i's as functions of the X_i's. Let us denote

$$\overset{\star}{x}(\mathbf{x}) = (\overset{\star}{x}_1(x_1), \ldots, \overset{\star}{x}_n(x_1, \ldots, x_n))$$

as the vector given by

$$\overset{\star}{x}_1(x_1) = F_{X_1}(x_1),$$
$$\overset{\star}{x}_i(x_1, \ldots, x_i) = F_{\left\{X_i | \bigcap_{j=1}^{i-1} X_j = x_j\right\}}(x_i) \quad \text{for } i = 2, \ldots, n.$$

It is also well known, see for instance Shaked and Shanthikumar (1998) that

$$\overset{\star}{x}(\mathbf{X}) =_{st} \mathbf{U}. \tag{6}$$

Let us consider the n-dimensional function $\phi(\mathbf{x}) = (\phi_1(x_1), \ldots, \phi_n(x_1, \ldots, x_n))$ defined as

$$\phi_1(x_1) = (\hat{y}_1 \circ \overset{\star}{x}_1)(x_1) = Q_{Y_1}(F_{X_1}(x_1)), \tag{7}$$

$$\phi_i(x_1, \ldots, x_i) = (\hat{y}_i \circ \overset{\star}{x}_i)(x_1, \ldots, x_i) = Q_{\left\{Y_i | \bigcap_{j=1}^{i-1} Y_j = \phi_j(x_1, \ldots, x_j)\right\}}\left(F_{\left\{X_i \bigcap_{j=1}^{i-1} X_j = x_j\right\}}(x_i)\right), \tag{8}$$

for $i = 2, \ldots, n$.

It is clear from (5) and (6) that ϕ maps stochastically $\mathbf{X}$ to $\mathbf{Y}$, that is $\phi(\mathbf{X}) =_{st} \mathbf{Y}$. It is also apparent that the standard construction can be seen as a generalization of the univariate quantile function.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors on $\mathbb{R}^n$ and let $\hat{x}(\cdot)$, $\hat{y}(\cdot)$ the corresponding standard constructions. Shaked and Shanthikumar (1998) considered the following condition:

$$\hat{y}(\mathbf{u}) - \hat{x}(\mathbf{u}) \text{ is increasing in } \mathbf{u} \in (0,1)^n, \tag{9}$$

as a multivariate generalization of (1). To maintain a coherent notation, we will say that $\mathbf{X}$ is less than $\mathbf{Y}$ in variability if (9) holds, and we will denote it as $\mathbf{X} \leqslant_{var} \mathbf{Y}$. Note that (9) can be rewritten in terms of the function ϕ defined in (7) and (8)

$$\phi(\hat{x}(\mathbf{u})) - \hat{x}(\mathbf{u}) \text{ is increasing in } \mathbf{u} \in (0,1)^n.$$

Another multivariate generalization based on the standard construction was given recently by Fernández-Ponce and Suárez-Llorens (2003). Given two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, we say that $\mathbf{X}$ is less in the *multivariate dispersive order* than $\mathbf{Y}$, denoted by $\mathbf{X} \leqslant_{disp} \mathbf{Y}$, if, and only, if

$$\|\hat{x}(\mathbf{v}) - \hat{x}(\mathbf{u})\|_2 \leqslant \|\hat{y}(\mathbf{v}) - \hat{y}(\mathbf{u})\|_2$$

for all $\mathbf{u}, \mathbf{v} \in [0,1]^n$, where $\|\cdot\|_2$ means the Euclidean distance. Note that the last inequality can be considered as a multivariate extension of (1). Also Fernández-Ponce and Suárez-Llorens (2003) proved that the $\leqslant_{disp}$ order is equivalent to check if the function ϕ defined in (7) and (8) is a multivariate expansion function, where multivariate expansion means that $||\phi(\mathbf{x}) - \phi(\mathbf{x}')||_2 \geqslant ||\mathbf{x} - \mathbf{x}'||_2$ for all $\mathbf{x}, \mathbf{x}'$ in $\mathbb{R}^n$, and therefore can be also seen as a multivariate generalization of (3).

3. The conditional dispersive order

In this section, we present a new multivariate dispersion order as a generalization of (3) for the univariate case. The purpose of this order is to relate the $\leqslant_{\text{disp}}$ and $\leqslant_{\text{var}}$ orders as we will see later on in Section 4. On the other hand, it will also give a meaningful interpretation in terms of dispersion to both orders.

Definition 1. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors. We consider the function ϕ defined in (7) and (8) which maps $\mathbf{X}$ onto $\mathbf{Y}$. We say that $\mathbf{X}$ is less in the conditional dispersive order than $\mathbf{Y}$, denoted by $\mathbf{X}\leqslant_{c-\text{disp}}\mathbf{Y}$, if $\phi_i(x_1,\ldots,x_i)$ is an expansion function in x_i in the univariate sense, for all $i=1,\ldots,n$.

Roughly speaking, the conditional dispersive order is defined through a function which maps stochastically a vector to another one and satisfies that ith component is an expansion function in x_i when $x_1,\ldots,x_{i-1}$ remains fixed. We observe that from (7), (8) and (3) we have that $\phi_1(x_1)$ is an expansion function in x_1 if, and only if,

$$X_1\leqslant_{\text{disp}}Y_1 \tag{10}$$

and for fixed $(x_1,\ldots,x_{i-1})\in\mathbb{R}^{i-1}$, $\phi_i(x_1,\ldots,x_i)$ is an expansion function in x_i, if, and only if,

$$\left\{X_i\left|\bigcap_{j=1}^{i-1}X_j=\hat{x}_j(u_1,\ldots,u_j)\right.\right\}\leqslant_{\text{disp}}\left\{Y_i\left|\bigcap_{j=1}^{i-1}Y_j=\hat{y}_j(u_1,\ldots,u_j)\right.\right\}, \tag{11}$$

for $i=2,\ldots,n$ and for all u_i such that $0<u_i<1$, $i=1,\ldots,n$.

Therefore the conditional dispersive order can be checked by the univariate dispersive order. The conditions Eqs. (10) and (11) provide a geometrical interpretation for the $\leqslant_{c-\text{disp}}$ order which we present in the bivariate case.

Fernández-Ponce and Suárez-Llorens (2003) provided the notion of corrected orthant associated with the standard construction and studied the accumulated probability in all of them. Let $\mathbf{X}=(X_1,X_2)$ be a bivariate random vector and let (u_1,u_2) in $[0,1]^2$. The intersection between the curve $x_2=\hat{x}_2(\cdot,u_2)$, where u_2 is fixed, and the line $x_1=\hat{x}_1(u_1)$ is achieved at the point $\hat{x}(u_1,u_2)=(\hat{x}_1(u_1),\hat{x}_2(u_1,u_2))$ and this intersection provides four corrected orthants (2^n on $\mathbb{R}^n$) which we represent in Fig. 1.

In order to clarify the understanding of the corrected orthant concept, we show the definition of the left lower corrected orthant one

$$\{(x_1,x_2):x_1\leqslant\hat{x}_1(u_1),x_2\leqslant\hat{x}_2(F_{X_1}(x_1),u_2)\}.$$

The result states that the accumulated probability in all corrected orthants depend on u_1 and u_2 (see Fig. 1). Note that the conditional function $x_2=\hat{x}_2(\cdot,u_2)$ represents the extreme behavior of the dependent variable X_2 conditional on the explanatory variable X_1.

Let us consider now (u_1,u_2) and $(v_1,v_2)\in[0,1]^2$ where $u_i<v_i$, $i=1,2$. According to the previous discussion is easy to see that Fig. 2 represents a bivariate central region for the random vector $\mathbf{X}$ given by the conditional function. Let consider now $\mathbf{Y}=(Y_1,Y_2)$ such that $\mathbf{X}\leqslant_{c-\text{disp}}\mathbf{Y}$. Then it holds by

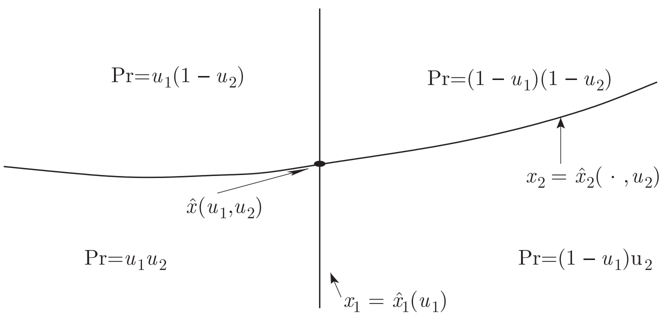

Fig. 1. The four corrected orthants and accumulated probabilities.

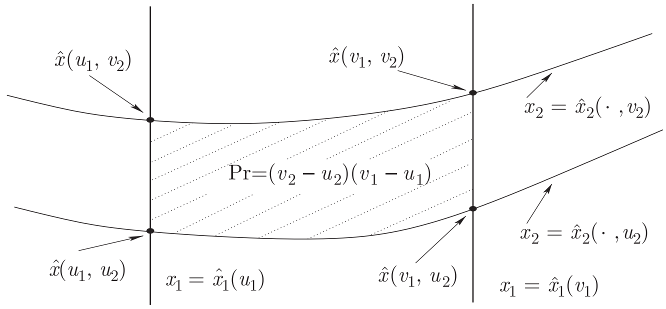

Fig. 2. Central region for **X**.

Eqs. (10) and (11) that

$$\hat{x}_1(v_1) - \hat{x}_1(u_1) \leqslant \hat{y}_1(v_1) - \hat{y}_1(v_1),$$
$$\hat{x}_2(u, v_2) - \hat{x}_2(u, u_2) \leqslant \hat{y}_2(u, v_2) - \hat{y}_2(u, u_2),$$

for all $0 \leqslant u_i \leqslant v_i \leqslant 1$, $i = 1, 2$, $u \in [0, 1]$. Then it is easy to note that the central region for **X** represented in Fig. 2 is less widely separated than the corresponding for **Y**. Therefore the interpretation of $\leqslant_{c-\text{disp}}$ in terms of dispersion is clear: we have found a wider region that accumulates the same probability.

To continue with the interpretation we observe that given an n-dimensional random vector **X** with density function $f_{\mathbf{X}}$ then

$$f_{\mathbf{X}}(\hat{x}(\mathbf{u})) = f_1(\hat{x}(u_1)) \prod_{i=2}^{n} f_{i|1,\ldots,i-1}(\hat{x}_i(u_1, \ldots, u_{i-1})),$$

for all $\mathbf{u} \in [0, 1]^n$, where f_1 is the density function of X_1 and $f_{i|1,\ldots,i-1}$ is the density function of $(X_i | \cap_{j=1}^{i-1} X_j = \hat{x}_j(u_1, \ldots, u_j))$. Therefore from Eqs. (10), (11) and (4), it is clear that if $\mathbf{X} \leqslant_{c-\text{disp}} \mathbf{Y}$ then

$$f_{\mathbf{X}}(\hat{x}(\mathbf{u})) \geqslant f_{\mathbf{Y}}(\hat{y}(\mathbf{u})) \quad \forall \mathbf{u} \in [0, 1]^n.$$

Hence the multivariate density functions evaluated at standard constructions are ordered as a clear generalization of (4).

We observe that $\leqslant_{c-\text{disp}}$ is not invariant under the permutation of the marginal distributions because the standard construction depends on the choice of the ordering of the marginal distributions. This is also the case of the $\leqslant_{\text{disp}}$ and $\leqslant_{\text{var}}$ orders. On the other hand it is easy to prove that $\leqslant_{c-\text{disp}}$ is location invariant.

The $\leqslant_{c-\text{disp}}$ order can be also interpreted in terms of local volume elements. Let **X** and **Y** two n-dimensional random vectors. Oja (1983) defined that **Y** is more scattered than **X**, denoted by $\mathbf{X} \leqslant_{\Delta} \mathbf{Y}$, if there is a function $k : \mathbb{R}^n \longmapsto \mathbb{R}^n$ such that $\mathbf{Y} =_{\text{st}} k(\mathbf{X})$ and for all $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\} \subset \mathbb{R}^n$ it holds that

$$\Delta(k(\mathbf{x}_1), \ldots, k(\mathbf{x}_{n+1})) \geqslant \Delta(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}), \tag{12}$$

where $\Delta(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})$ is the volume of the "simplex" with vertices at $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$.

Note that although condition (12) seems to be a strict one, in practice, it is only necessary to check if the absolute value of the determinant of the Jacobian matrix of k is bigger than 1 (see Giovagnoli and Wynn, 1995), i.e.

$$\text{abs}(\text{Det}(J_k(\mathbf{x}))) \geqslant 1 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n. \tag{13}$$

Let us consider two n-dimensional random vectors **X** and **Y** such that $\mathbf{X} \leqslant_{c-disp} \mathbf{Y}$. From the characterization of the univariate expansion functions it easily holds that $\partial \phi_i / \partial x_i \geqslant 1$ for $i = 1, \ldots, n$. Due to the fact that ϕ has a lower triangular Jacobian matrix it is apparent that $\text{Det}(J_\phi) \geqslant 1$. Hence from (13) it holds that $\leqslant_{c-\text{disp}} \Rightarrow \leqslant_{\Delta}$.

Fernández-Ponce and Suárez-Llorens (2003) proved in Theorem 3.1 that if we take a function k such that $\mathbf{Y} =_{\text{st}} k(\mathbf{X})$ and k has a lower triangular matrix with diagonal elements strictly positive then k has the form of the function ϕ. From this result there are many possible distributions which can be compared in the $\leqslant_{c-\text{disp}}$

ordering. If we consider a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ and a transformation $\mathbf{Y} = \phi(\mathbf{X})$, such that $Y_i = f_i(X_i) + g_i(X_1, \ldots, X_{i-1})$, where f_i is an strictly increasing differentiable expansion function and g_i is any differentiable function for all $i = 1, \ldots, n$, then it is apparent that $\mathbf{X} \leqslant_{c-\mathrm{disp}} \mathbf{Y}$. As a particular case, this result holds for $f_i(x) = a_i x$ where $a_i > 1$, for $i = 1, \ldots, n$.

To finish this section we present a result for the preservation of the conditional dispersive order under conjunction of independent random vectors ordered in the conditional dispersive order.

Theorem 1. *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$ be a set of independent random vectors where the dimension of $\mathbf{X}_i$ is n_i, $i = 1, 2, \ldots, m$ and let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_m$ be another set of independent random vectors where the dimension of $\mathbf{Y}_i$ is n_i, $i = 1, 2, \ldots, m$. If $\mathbf{X}_i \leqslant_{c-\mathrm{disp}} \mathbf{Y}_i$ for $i = 1, 2, \ldots, m$ then*

$$(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_m) \leqslant_{c-\mathrm{disp}} (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_m).$$

4. The relationships among $\leqslant_{\mathrm{disp}}$, $\leqslant_{\mathrm{var}}$ and $\leqslant_{c-\mathrm{disp}}$ orders

In this section we study the relationships among the $\leqslant_{\mathrm{disp}}$, $\leqslant_{\mathrm{var}}$ and $\leqslant_{c-\mathrm{disp}}$ orders. The relationship among the $\leqslant_{\mathrm{disp}}$ and conditions Eqs. (10) and (11) was given by Fernández-Ponce and Suárez-Llorens (2003) (see Corollary 3.1), and therefore the following result holds.

Theorem 2. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors. If $\mathbf{X} \leqslant_{\mathrm{disp}} \mathbf{Y}$ then $\mathbf{X} \leqslant_{c-\mathrm{disp}} \mathbf{Y}$.*

We want to emphasize that Fernández-Ponce and Suárez-Llorens (2003) did not consider Eqs. (10) and (11) as a multivariate dispersive order and they just studied the relationship among these conditions and the $\leqslant_{\mathrm{disp}}$ order.

Next we show that the variability order is stronger than the conditional dispersive order.

Theorem 3. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors. If $\mathbf{X} \leqslant_{\mathrm{var}} \mathbf{Y}$ then $\mathbf{X} \leqslant_{c-\mathrm{disp}} \mathbf{Y}$.*

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors and let $\hat{x}(\cdot)$, $\hat{y}(\cdot)$ the corresponding standard constructions. In particular, condition (9) implies that

$$\hat{y}_i(u_1, u_2, \ldots, u_i) - \hat{x}_i(u_1, u_2, \ldots, u_i) \text{ is increasing in } u_i \in (0,1) \quad \text{for } i = 1, 2, \ldots, n.$$

Note that if we fix $u_1, \ldots, u_{i-1}$ then $\hat{y}_i(u_1, \ldots, u_i)$ and $\hat{x}_n(u_1, \ldots, u_i)$ are the univariate quantile functions of $(X_i | \cap_{j=1}^{i-1} X_j = \hat{x}_j(u_1, \ldots, u_j))$ and $(Y_i | \cap_{j=1}^{i-1} Y_j = \hat{y}_j(u_1, \ldots, u_j))$, respectively. Hence the proof follows from Eqs. (10) and (11). □

Next we show that previous implications are strict and also we show the $\leqslant_{\mathrm{disp}}$ and $\leqslant_{\mathrm{var}}$ orders, in general, are not related.

Example 1. Let $\mathbf{X} \sim \mathrm{N}((0,0), \Sigma_{\mathbf{X}})$ and $\mathbf{Y} \sim \mathrm{N}((0,0), \Sigma_{\mathbf{Y}})$ be two bivariate normal distributions where

$$\Sigma_{\mathbf{X}} = \begin{pmatrix} \sigma_{1\mathbf{X}} & \sigma_{12\mathbf{X}} \\ \sigma_{12\mathbf{X}} & \sigma_{2\mathbf{X}} \end{pmatrix} \quad \text{and} \quad \Sigma_{\mathbf{Y}} = \begin{pmatrix} \sigma_{1\mathbf{Y}} & \sigma_{12\mathbf{Y}} \\ \sigma_{12\mathbf{Y}} & \sigma_{2\mathbf{Y}} \end{pmatrix}.$$

Note that without lack of generality the mean vectors can be considered with components equal to zero. It is well known that the conditional distributions for multivariate normal distribution are also normal distributions. Then, the distribution of X_2 conditioned to X_1 is given by

$$(X_2|_{X_1 = x_1}) \sim \mathrm{N}\left(x_1 \frac{\sigma_{12\mathbf{X}}}{\sigma_{1\mathbf{X}}^2}, \sigma_{2\mathbf{X}} \sqrt{(1 - \rho_{\mathbf{X}}^2)} \right),$$

where $\rho_{\mathbf{X}}$ represents the correlation coefficient of X_1 and X_2. Analogously for Y_2 conditioned to Y_1.

First, we consider the $\leqslant_{c-\mathrm{disp}}$ ordering. It is a well known result that two univariate normal distributions are ordered in the univariate dispersion sense if, and only if their variances are ordered. Hence using Eqs. (10)

and (11) it easily holds that $\mathbf{X}\leqslant_{c-\text{disp}}\mathbf{Y}$ if, and only if

$$\sigma_{1\mathbf{Y}}\geqslant\sigma_{1\mathbf{X}},$$
$$\sigma_{2\mathbf{Y}}\sqrt{1-\rho_{\mathbf{Y}}^2}\geqslant\sigma_{2\mathbf{X}}\sqrt{1-\rho_{\mathbf{X}}^2}. \tag{14}$$

Now we consider the $\leqslant_{\text{disp}}$ order. Fernández-Ponce and Suárez-Llorens (2003) studied the Jacobian matrix of the function ϕ given in (7) and (8) for the bivariate normal distribution and they obtained a 2×2 lower triangular matrix of the form

$$J_\phi=\begin{pmatrix}\frac{\sigma_{1\mathbf{Y}}}{\sigma_{1\mathbf{X}}} & 0\\ \frac{\sigma_{2\mathbf{Y}}}{\sigma_{1\mathbf{X}}}\left(\rho_{\mathbf{Y}}-\rho_{\mathbf{X}}\frac{\sqrt{(1-\rho_{\mathbf{Y}}^2)}}{\sqrt{(1-\rho_{\mathbf{X}}^2)}}\right) & \frac{\sigma_{2\mathbf{Y}}\sqrt{(1-\rho_{\mathbf{Y}}^2)}}{\sigma_{2\mathbf{X}}\sqrt{(1-\rho_{\mathbf{X}}^2)}}\end{pmatrix}.$$

Giovagnoli and Wynn (1995) characterized a multivariate expansion function in terms of its Jacobian matrix. Let $k(\cdot)$ be a continuously differentiable function. Then $k(\cdot)$ is a multivariate expansion function if, and only if the matrix $J_k(\mathbf{x})^tJ_k(\mathbf{x})-I_n$ is nonnegative definite $\forall\mathbf{x}\in\mathbb{R}^n$, where $J_k=\{\partial k_i/\partial x_j\}$ is the Jacobian matrix of $k(\cdot)$ and I_n is the identity matrix of order n. Under a straightforward computation of the eigenvalues of the matrix $J_\phi(\mathbf{x})^tJ_\phi(\mathbf{x})-I_2$ we obtain that this one is nonnegative definite if, and only if the inequalities given by (14) hold and

$$\sigma_{2\mathbf{Y}}^2\left(\rho_{\mathbf{Y}}\sqrt{1-\rho_{\mathbf{X}}^2}-\rho_{\mathbf{X}}\sqrt{1-\rho_{\mathbf{Y}}^2}\right)^2\leqslant(\sigma_{1\mathbf{Y}}^2-\sigma_{1\mathbf{X}}^2)(\sigma_{2\mathbf{Y}}^2(1-\rho_{\mathbf{Y}}^2)-\sigma_{2\mathbf{X}}^2(1-\rho_{\mathbf{X}}^2)). \tag{15}$$

Let us consider now the $\leqslant_{\text{var}}$ order. Given that we deal with conditional normal distributions, the standard construction for $\mathbf{X}$ is easily given by $\hat{x}_1(u_1)=Q_Z(u_1)\sigma_{1\mathbf{X}}$ and $\hat{x}_2(u_1,u_2)=Q_Z(u_1)\rho_{\mathbf{X}}\sigma_{2\mathbf{X}}+Q_Z(u_2)\sigma_{2\mathbf{X}}\sqrt{(1-\rho_{\mathbf{X}}^2)}$ where Z is distributed as $N(0,1)$ and analogously for $\mathbf{Y}$. Then $\mathbf{X}\leqslant_{\text{var}}\mathbf{Y}$ holds if, and only if the inequalities given by (14) hold and

$$\sigma_{2\mathbf{Y}}\rho_{\mathbf{Y}}\geqslant\sigma_{2\mathbf{X}}\rho_{\mathbf{X}}. \tag{16}$$

Note that from (16) the $\leqslant_{\text{var}}$ ordering depends on the signs of the correlation coefficients.

Now we present some detailed cases in order to study the strict implications among the different dispersion orders. If we take $\sigma_{i\mathbf{X}}=\sigma_{i\mathbf{Y}}$, for $i=1,2$, then (14) holds if, and only if $|\rho_{\mathbf{X}}|\geqslant|\rho_{\mathbf{Y}}|$. If we take $\rho_{\mathbf{X}}=-0,7$ and $\rho_{\mathbf{Y}}=0,4$, then $\mathbf{X}\leqslant_{c-\text{disp}}\mathbf{Y}$ holds and from (16) $\mathbf{X}\leqslant_{\text{var}}\mathbf{Y}$ also holds. However from (15) $\mathbf{X}\leqslant_{\text{disp}}\mathbf{Y}$ does not hold. Under the same assumption for the marginal variances and taking $\rho_{\mathbf{X}}=0,7$ and $\rho_{\mathbf{Y}}=0,4$, $\mathbf{X}\leqslant_{c-\text{disp}}\mathbf{Y}$ holds. However from (15) and (16) neither $\mathbf{X}\leqslant_{\text{disp}}\mathbf{Y}$ nor $\mathbf{X}\leqslant_{\text{var}}\mathbf{Y}$ hold, respectively. On the other hand if we take $\rho_{\mathbf{X}}=\rho_{\mathbf{Y}}$ and $\sigma_{i\mathbf{X}}<\sigma_{i\mathbf{Y}}$, for $i=1,2$. Then from (14) and (15), both $\mathbf{X}\leqslant_{c-\text{disp}}\mathbf{Y}$ and $\mathbf{X}\leqslant_{\text{disp}}\mathbf{Y}$ hold, respectively. However the $\leqslant_{\text{var}}$ ordering depends on the signs of the correlation coefficients. In particular if we consider $\rho_{\mathbf{X}}=\rho_{\mathbf{Y}}<0$ does not hold.

To summarize we have the following chains of strict implications:

$$\begin{array}{ccccc}\leqslant_{\text{disp}} & \Rightarrow & \leqslant_{c-\text{disp}} & \Rightarrow & \leqslant_{\Delta}\\ & & \Uparrow & & \\ & & \leqslant_{\text{var}}. & & \end{array}$$

Following Müller and Scarsini (2001), Khaledi and Kochar (2005), and Arias-Nicolás et al. (2005), a natural question arises: Whether the $\leqslant_{c-\text{disp}}$ order holds under the dispersive order of the marginals, for random vectors with the same copula?

A copula C is a cumulative distribution function with uniform margins on $[0,1]$. Furthermore, it has been shown that if H is a n-dimensional distribution function, with marginal distribution functions $F_1,\ldots,F_n$ then there exists a n-copula C such that for all $(x_1,\ldots,x_n)\in\mathbb{R}^n$, it holds that $H(x_1,\ldots,x_n)=C(F_1(x_1),\ldots,F_n(x_n))$. Moreover, if $F_1,\ldots,F_n$ are continuous then C is unique (for details about copulas see Nelsen, 1999). It follows that if $\mathbf{X}=(X_1,\ldots,X_n)$ and $\mathbf{Y}=(Y_1,\ldots,Y_n)$ are two n-dimensional random variables, then they have the same copula if, and only if $(X_1,\ldots,X_n) =_{\mathrm{st}} (Q_{X_1}(F_{Y_1}(Y_1)),\ldots,Q_{X_n}(F_{Y_n}(Y_n)))$. To summarize the copula allows us to separate the effect of the dependence from effects of the marginal distributions. With this settings, we can formulate the following theorem.

Theorem 4. *Let* $\mathbf{X} = (X_1,\ldots,X_n)$ *and* $\mathbf{Y} = (Y_1,\ldots,Y_n)$ *be two n-dimensional random vectors such that they have the same copula. Then* $\mathbf{X} \leqslant_{c-\mathrm{disp}} \mathbf{Y}$ *if and only if* $X_i \leqslant_{\mathrm{disp}} Y_i$ *for all* $i = 1,\ldots,n$.

Proof. Arias-Nicolás et al. (2005) showed that, for two random vectors with the same copula, the function ϕ can be expressed as

$$\phi_i(x_1,\ldots,x_i) = Q_{Y_i}(F_{X_i}(x_i)), \tag{17}$$

for $i = 1,\ldots,n$. Hence in light of (3) the result holds. □

Therefore given two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, with the same copula, then

$$\begin{array}{ccccc} \mathbf{X} \leqslant_{\mathrm{disp}} \mathbf{Y} & \Leftrightarrow & \mathbf{X} \leqslant_{c-\mathrm{disp}} \mathbf{Y} & \Leftrightarrow & X_i \leqslant_{\mathrm{disp}} Y_i \quad \forall i = 1,2,\ldots,n. \\ & & \Uparrow & & \\ & & \mathbf{X} \leqslant_{\mathrm{var}} \mathbf{Y}. & & \end{array} \tag{18}$$

Looking at the last discussion of the Example 1 and from the well known fact that two bivariate normal distributions have a common copula if, and only if they have the same correlation coefficient then it is apparent a strict implication for $\leqslant_{\mathrm{var}}$ holds. As a special case of distribution functions having the same structure dependence are two random vectors $\mathbf{X}$ and $\mathbf{Y}$ having independent components.

The strict implication in (18) for the $\leqslant_{\mathrm{var}}$ ordering is not surprising if we take into account that the $\leqslant_{\mathrm{var}}$ ordering seems to be associated with the positive dependence structure of the random vectors. Those considerations lead us to assume some dependence properties for the random vectors. To simplify the notation we will say that the random vector $(X_1,\ldots,X_n)$ is *conditionally increasing in quantile*, denoted by CIQ in order to simplify, if the standard construction $\hat{x}(\mathbf{u})$ is increasing in $\mathbf{u} \in (0,1)^n$.

The CIQ notion is related to the following property. The random vector $(X_1,\ldots,X_n)$ is said to be *conditionally increasing in sequence* (CIS) if (see Barlow and Proschan, 1975), $X_i \uparrow_{\mathrm{st}} (X_1,\ldots,X_{i-1})$, $i = 2,\ldots,n$, that is, if

$$\begin{aligned} &[X_i | X_1 = x_1,\ldots,X_{i-1} = x_{i-1}] \leqslant_{\mathrm{st}} [X_i | X_1 = x'_1,\ldots,X_{i-1} = x'_{i-1}] \\ &\quad \text{whenever } x_j \leqslant x'_j, \quad j = 1,2,\ldots,i-1. \end{aligned}$$

Rubinstein et al. (1985) proved that the CIS property implies the CIQ property. Also the CIQ property is preserved by strictly increasing transformations of each component.

Theorem 5. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two n-dimensional random vectors with the same copula. If* $\mathbf{X}$ *is CIQ then* $\mathbf{X} \leqslant_{\mathrm{var}} \mathbf{Y}$ *if, and only if* $X_i \leqslant_{\mathrm{disp}} Y_i$ *for all* $i = 1,\ldots,n$.

Proof. Note that from (18) it is only necessary to show the sufficient condition. Under the expression of ϕ for random vectors with the same copula given by Eq. (17) and the fact that ϕ maps the standard construction of $\mathbf{X}$ to the corresponding of $\mathbf{Y}$ it easily holds that $Q_{Y_i}(F_{X_i}(\hat{x}_i(u_1,\ldots,u_i)) = \hat{y}_i(u_1,\ldots,u_i)$ for all $i = 1,\ldots,n$. Hence

$$F_{X_i}(\hat{x}_i(u_1,\ldots,u_i)) = F_{Y_i}(\hat{y}_i(u_1,\ldots,u_i)) \quad \text{for all } i = 1,\ldots,n. \tag{19}$$

From (19) if $\mathbf{X}$ is CIQ is apparent that $\mathbf{Y}$ is also CIQ. If we take $(v_1,\ldots,v_i) \geqslant (u_1,\ldots,u_i)$ then

$$\begin{aligned} &\hat{y}_i(v_1,\ldots,v_i) \geqslant \hat{y}_i(u_1,\ldots,u_i), \\ &\hat{x}_i(v_1,\ldots,v_i) \geqslant \hat{x}_i(u_1,\ldots,u_i), \end{aligned}$$

for $i = 1,\ldots,n$. On the other hand, from (19) it holds that $\hat{x}_i(u_1,\ldots,u_i)$ and $\hat{y}_i(u_1,\ldots,u_i)$ provide the same univariate quantile for the marginal variables X_i and Y_i, respectively. By hypothesis assumption the marginal distributions are ordered in dispersion. Then

$$\hat{y}_i(v_1,\dots,v_i) - \hat{y}_i(u_1,\dots,u_i) \geqslant \hat{x}_i(v_1,\dots,v_i) - \hat{x}_i(u_1,\dots,u_i),$$

for all $(v_1,\dots,v_i) \geqslant (u_1,\dots,u_i)$. Then we obtain condition (9). Hence $\mathbf{X} \leqslant_{\text{var}} \mathbf{Y}$. □

Therefore from (18) and Theorem 5, given two random vectors $\mathbf{X}$ and $\mathbf{Y}$, with the same copula and with the CIQ property, then we have the following chain of equivalences:

$$\mathbf{X} \leqslant_{\text{disp}} \mathbf{Y} \;\Leftrightarrow \mathbf{X} \;\leqslant_{c-\text{disp}} \mathbf{Y} \;\Leftrightarrow\; \mathbf{X} \leqslant_{\text{var}} \mathbf{Y} \;\Leftrightarrow\; X_i \leqslant_{\text{disp}} Y_i \quad \forall i = 1,\dots,n. \tag{20}$$

We would like to emphasize that the equivalences given by (20) simplify the conditions where the $\leqslant_{\text{var}}$ ordering can be verified which, from a practical viewpoint, can be very complicated to compute. Specifically, we can take advantage of (20) in the applications of the $\leqslant_{\text{var}}$ ordering.

5. Applications

5.1. Models of ordered random variables

An interesting application of the implications given by (20) can be given for several models of ordered random variables with applications in statistics and reliability. For instance, the order statistics from a sample of i.i.d. random variables, the random vector of the first n epoch times of a nonhomogeneous Poisson process, k records and order statistics under multivariate imperfect repair. A general concept where the previous models are included is the concept of generalized order statistics see Kamps (1995). Formally, let $n \in \mathbb{N}$, $k \geqslant 1$, $m_1,\dots,m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leqslant r \leqslant n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geqslant 1$ for all $r \in 1,\dots,n-1$, and let $\tilde{m} = (m_1,\dots,m_{n-1})$, if $n \geqslant 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary, if $n = 1$). We call uniform generalized order statistics to the random vector $(U_{(1,n,\tilde{m},k)},\dots,U_{(n,n,\tilde{m},k)})$ with joint density function

$$h(u_1,\dots,u_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1}(1-u_j)^{m_j}\right)(1-u_n)^{k-1}$$

on the cone $0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant 1$. Now given a distribution function F we call generalized order statistics (GOS) based on F to the random vector

$$(X_{(1,n,\tilde{m},k)},\dots,X_{(n,n,\tilde{m},k)}) \equiv (F^{-1}(U_{(1,n,\tilde{m},k)}),\dots,F^{-1}(U_{(n,n,\tilde{m},k)})). \tag{21}$$

Note that if F is an absolutely continuous distribution then F^{-1} is strictly increasing. Therefore it is apparent from (21) that two random vectors of GOS with the same parameters, and possibly, based on different absolutely continuous distributions F and G, have the same copula. In addition, a random vector of GOS has the CIS property (this follows from the Markovian property of GOS and the transition probabilities), so it also has the CIQ property.

Under the previous arguments and using (20) it holds that two random vectors of GOS $(X_{(1,n,\tilde{m},k)},\dots,X_{(n,n,\tilde{m},k)})$ and $(Y_{(1,n,\tilde{m},k)},\dots,Y_{(n,n,\tilde{m},k)})$, based on absolutely continuous distribution functions F and G, respectively, are ordered in the $\leqslant_{\text{disp}}$, $\leqslant_{c-\text{disp}}$ and $\leqslant_{\text{var}}$ multivariate dispersive orderings if, and only if the marginals are ordered in the univariate dispersive order, that is if $X_{(r,n,\tilde{m},k)} \leqslant_{\text{disp}} Y_{(r,n,\tilde{m},k)}$, for all $r : 1,\dots,n$. The last condition holds under the assumption $F \leqslant_{\text{disp}} G$ (see Belzunce et al. (2005), Theorem 3.12). With a simpler argument, this result simplifies Theorem 3.11 by Belzunce et al. (2005), Theorem 2.3 by Belzunce et al. (2003) and Theorem 3.1 by Belzunce and Ruiz (2002).

5.2. Variability ordering of convolutions

Convolutions appear naturally in several context such as risk theory, reliability and statistics. For example consider an insurance company with a number n of clients, with individual risks $X_1,\dots,X_n$ then the company bears the risk $S = \sum_{i=1}^{n} X_i$. In reliability theory convolutions appear when a failed unit is replaced by a new one and the total life is obtained by the addition of the two lifelength. Also several statistics are linear combinations of random variables. In the literature one can find several results on variability comparisons of convolutions. Most of these results are given for some parametric models (uniform and gamma distributions) of independent random variables see for example Kochar and Ma (1999a, b), Korwar (2002), and Khaledi and Kochar (2002, 2004) in the case of the dispersive order. For dependent components a simple but elegant result is provided by Bäuerle (1997), Bäuerle and Rieder (1997) and Müller (1997), which prove that the convolution of the components of two random vectors, ordered in the supermodular order, are ordered in the increasing convex order.

One of the most interesting properties of the $\leqslant_{\text{var}}$ ordering is given by the study of conditions for the variability ordering of increasing directionally transformations of the random vectors. Recall from Rüschendorff (1983) that a real function φ on $\mathbb{R}^n$ is said to be *directionally convex* if for any $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1, 2, 3, 4$, such that $\mathbf{x}_1 \leqslant \mathbf{x}_2 \leqslant \mathbf{x}_4$, $\mathbf{x}_1 \leqslant \mathbf{x}_3 \leqslant \mathbf{x}_4$ and $\mathbf{x}_1 + \mathbf{x}_4 = \mathbf{x}_2 + \mathbf{x}_3$, one has

$$\varphi(\mathbf{x}_2) + \varphi(\mathbf{x}_3) \leqslant \varphi(\mathbf{x}_1) + \varphi(\mathbf{x}_4). \tag{22}$$

Shaked and Shanthikumar (1998) in Theorem 4.2 proved that given two nonnegative n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, with the CIS property, if $\mathbf{X} \leqslant_{\text{var}} \mathbf{Y}$ then $\varphi(\mathbf{X}) \leqslant_{\text{st:icx}} \varphi(\mathbf{Y})$, for all increasing directionally convex functions φ. Where the $\leqslant_{\text{st:icx}}$ ordering means that $E[h(\varphi(\mathbf{X}))] \leqslant E[h(\varphi(\mathbf{Y}))]$ for all increasing functions h for which the expectations exists (that is, if $\varphi(\mathbf{X}) \leqslant_{\text{st}} \varphi(\mathbf{Y})$) and $\text{Var}[h(\varphi(\mathbf{X}))] \leqslant \text{Var}[h(\varphi(\mathbf{Y}))]$ for all increasing convex functions h for which the variances exist. The $\leqslant_{\text{st:icx}}$ ordering is of interest because it lets you to compare the variances.

Note that Shaked and Shanthikumar (1998) assume the $\leqslant_{\text{var}}$ ordering between two nonnegative random vectors and the CIS property. If we consider random vectors with a common copula, using Theorem 5 and the mentioned fact that CIS implies CIQ, in this case the $\leqslant_{\text{var}}$ ordering can be easily checked just comparing in dispersion the marginal distributions. Hence from the fact that the function $\varphi(\mathbf{x}) = \sum_{i=1}^n x_i$ is increasing directionally convex we can state the following general result.

Corollary 1. *Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two n-dimensional nonnegative CIS random vectors with a common copula. If $X_i \leqslant_{\text{disp}} Y_i$ for every $i = 1, \ldots, n$, then $\sum_{i=1}^n X_i \leqslant_{\text{st:icx}} \sum_{i=1}^n Y_i$.*

In particular we have that $Var[h(\sum_{i=1}^n X_i)] \leqslant Var[h(\sum_{i=1}^n Y_i)]$, for every increasing convex function h, provided the previous variances exist, and therefore we can compare the variability of convolutions of random variables not necessarily independent.

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