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New potential symmetries for some evolution equations

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Abstract

In this paper we derive new potential symmetries that seem not to be recorded in the literature. These potential symmetries are determined by considering a generalized potential system, rather than the natural potential system or a general integral variable. An inhomogeneous diffusion equation, a porous medium equation and the Fokker–Planck equation have been considered as application of this procedure.

Keywords: Nonlocal symmetries; Potential symmetries

1. Introduction

It is well known that there exists partial differential equations (PDE's) of physical interest possessing few symmetries or none at all [14]. It turns out that PDE's can admit nonlocal symmetries whose infinitesimal generators depend on integrals of the dependent variables in some specific manner. The case for nonlocal symmetries goes further than this: anyway they can be used to generate highly nontrivial explicit solutions to PDE's (which in turn can have physical interpretation or can be useful for checking numerical schemes [12,17,18]), they appear in the theory of recursions operators and mastersymmetries, and they are related to Bäcklund and linearizing transformations. This last property was observed by Bluman, although the work of A.M. Vinogradov, and I.S. Krasil'shchik [22] should not be ignored.

Krasil'shchik and Vinogradov [10,11,22] gave criteria which must be satisfied by nonlocal symmetries of a PDE when realized as local symmetries of a system of PDE's which 'covers' the given PDE. Akhatov, Gazizov and Ibragimov [2] gave nontrivial examples of nonlocal symmetries generated by heuristic procedures. Nonlocal symmetries have been also studied in Refs. [1,4].

In Ref. [5] Bluman introduced a method to find a new class of symmetries for a PDE. Suppose a given scalar PDE of second order

$$F(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0, \tag{1}$$

where the subscripts denote the partial derivatives of u, can be written as a conservation law

$$\frac{\mathrm{D}}{\mathrm{D}t} f(x, t, u, u_x, u_t) - \frac{\mathrm{D}}{\mathrm{D}x} g(x, t, u, u_x, u_t) = 0, \tag{2}$$

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for some functions f and g of the indicated arguments. Here $\frac{D}{Dx}$ and $\frac{D}{Dt}$ are total derivative operators defined by

$$\frac{D}{Dx} = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots,$$
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots.$$

Through the conservation law (2) one can introduce an auxiliary potential variable v and form an auxiliary potential system (system approach)

$$v_x = f(x, t, u, u_x, u_t), \tag{3}$$
$$v_t = g(x, t, u, u_x, u_t). \tag{4}$$

For many physical equations one can eliminate u from the potential system (3) and form an auxiliary integrated or potential equation (integrated equation approach)

$$G(x, t, v, v_x, v_t, v_{xx}, v_{xt}, v_{tt}) = 0, \tag{5}$$

for some function G of the indicated arguments. Any Lie group of point transformations

$$\mathbf{v} = \xi(x, t, u, v)\frac{\partial}{\partial x} + \tau(x, t, u, v)\frac{\partial}{\partial t} + \phi(x, t, u, v)\frac{\partial}{\partial u} + \psi(x, t, u, v)\frac{\partial}{\partial v}, \tag{6}$$

admitted by (3) yields a nonlocal symmetry *potential symmetry* of the given PDE (2) if and only if the following condition is satisfied

$$\xi_v^2 + \tau_v^2 + \phi_v^2 \neq 0. \tag{7}$$

We point out that if we consider a Lie group of point transformations

$$\mathbf{w} = \xi(x, t, v)\frac{\partial}{\partial x} + \tau(x, t, v)\frac{\partial}{\partial t} + \psi(x, t, v)\frac{\partial}{\partial v} \tag{8}$$

admitted by (5) the condition

$$\xi_v^2 + \tau_v^2 \neq 0 \tag{9}$$

is a *sufficient* but not a *necessary* condition in order to yield nonlocal symmetries of (2). In Ref. [21] potential symmetries of a simplified model for reacting mixtures were derived as well as an invertible mapping which linearizes the reacting mixture model.

Although of course there are some classifications of nonlocal symmetries for some important equations such as KdV and Burgers [22], and it is well known that if a given PDE system has n conservation laws, then each conservation law yields potential equations and a corresponding nonlocally related potential system, generally speaking there is no hope of obtaining all the nonlocal symmetries of a given equation, in sharp contradiction with the local case. All one can hope for is classifying nonlocal symmetries of a given type or, more rigorously stated, all one can hope for is classifying nonlocal symmetries defined on a given covering of the equation at hand.

The method introduced by Bluman in Ref. [3] only requires the PDE to be written in a conserved form (2). Nevertheless, in most papers concerning evolution equations of the form

$$u_t = \frac{D}{D_x} g(x, t, u, u_x), \tag{10}$$

the auxiliary system considered is the so-called *natural* potential system

$$\begin{aligned} v_x &= u, \\ v_t &= g(x, t, u, u_x), \end{aligned} \tag{11}$$

and a different system is searched only when the given equation is not in a conserved form.

In Ref. [13] the authors developed a method in order to find potential symmetries which are missed by standard potential symmetry analysis. These symmetries were determined by using a general integral variable.

The aim of this paper is to develop a procedure to find hidden potential symmetries, which generalizes the method introduced in Ref. [13]. By using this method we find potential symmetries including those that are missed by standard potential symmetry analysis as well as a symmetry missed by introducing a general weighted integral variable in Ref. [13].

We present some nontrivial examples of nonlocal symmetries. This is interesting, since in a nonlocal setting one cannot use algorithmic methods as in the local symmetry case treated in Ref. [14].

The new procedure proposed in this paper is the following: instead of considering the natural potential system or the general integral variable we consider a general potential system

$$\begin{aligned} v_x &= f_1(x)h_1(u), \\ v_t &= f_2(x)h_2(u)u_x + f_3(x)h_3(u). \end{aligned} \tag{12}$$

By requiring the governing PDE (10) to be equivalent to the potential system (12) we get that functions $f_i(x)$ and $h_i(u) i = 1, 2, 3$ must satisfy several conditions that lead to a *hidden potential system*. A point symmetry (6) admitted by this system yields a *hidden potential symmetry* of the given PDE (2) if this symmetry is not a potential symmetry derived from the *natural* potential system and condition (7) is satisfied.

By showing several examples I wish to suggest the readers a procedure to obtain potential symmetries which *are missed* by standard potential symmetry analysis and *are missed* by considering a general integral variable. These examples are: an inhomogeneous diffusion equation, a porous medium equation and a Fokker–Planck equation. These three equations have been respectively considered in Refs. [19,13,6,16]. We show that this procedure yields new hidden potential symmetries for these equations, which as far as we know do not appear in the literature.

2. Inhomogeneous diffusion equation

One of the mathematical models for diffusion processes is the generalized inhomogeneous nonlinear diffusion equation

$$f(x)u_t = [g(x)u^n u_x]_x. \tag{13}$$

The diffusion processes appear in many physics processes such as plasma physics, kinetic theory of gases, solid state, metallurgy and transport in porous medium.

In (13) $u(x, t)$ is a function of position x and time t and may represent the temperature, $f(x)$ and $g(x)$ are arbitrary smooth functions of position and may denote the density and the density-dependent part of thermal diffusion, respectively.

In Ref. [19], C. Sophocleous has classified the potential symmetries of (13). He proved that potential symmetries exist only if the parameter n takes the values -2 or $-\frac{2}{3}$ and $f(x)g(x) = \textit{constant}$ or $g(x) = \frac{1}{f(x)}[\int f(x)\mathrm{d}x]^4$. It was pointed out in Ref. [19] that Eq. (13) is equivalent by means of a point transformation to

$$u_t = [g(x)u^n u_x]_x. \tag{14}$$

Consequently, by considering the natural potential system C. Sophocleous has derived potential symmetries, for

$$u_t = [g(x)u^{-2} u_x]_x, \tag{15}$$

only when $g(x) = \textit{constant}$ or $g(x) = x^4$. In Ref. [13] hidden nonlocal symmetries were determined by considering an integrated equation, obtained, using a general integral variable

$$\phi = \int k(x)u(x, t)\mathrm{d}x + J(t).$$

In Ref. [13] the authors found extra potential symmetries of (15) for $g(x) = x^2$ that corresponds to the case in which (15) is linearizable.

We are considering the more general auxiliary system (12) and we require (15) to be expressed in a conserved form as this system, leading to the following conditions

$$\begin{aligned} &h_1(u) = k_1 u + k_2, \qquad f_1(x) = \frac{k_3 k_4}{k_1} \int \frac{\mathrm{d}x}{g}, \\ &h_2(u) = \frac{k_1}{u^2}, \qquad f_2(x) = f_1 g, \\ &h_3(u) = \frac{k_4}{u} + k_5, \qquad f_3(x) = k_3. \end{aligned} \tag{16}$$

By setting, without loss of generality, $k_1 = 1$, $k_4 = 1$, $f_1(x) = k(x)$, $k_3 = -a$, and $k_2 = 0$ system (12) becomes

$$\begin{aligned} &v_x = k(x)u, \\ &v_t = k(x)g(x)\frac{u_x}{u^2} - \frac{a}{u}, \end{aligned} \tag{17}$$

with $a = constant$ and

$$k'(x)g(x) + a = 0. \tag{18}$$

If system (17) is invariant under a Lie group of point transformations with infinitesimal generator (6), then

$$\begin{aligned} &\xi = \xi(x,t,v), \qquad \tau = \tau(t), \qquad \psi = \psi(t,v), \\ &\phi = -k\xi_v u^2 + (\psi_v - \xi\frac{k'}{k} - \xi_x)u, \end{aligned} \tag{19}$$

where ξ, τ, ψ and k must satisfy the following equations

$$\begin{aligned} &gk(2\psi_v - \tau_t) - 2\xi gk' - \xi g'k = 0, \\ &\xi_{vv} gk^2 - \xi_t = 0, \\ &-gk^2(\psi_{vv} - 2\xi_{vx}) + \psi_t + 2\xi_v gkk' + 2a\xi_v k = 0, \\ &2ak\psi_v - \xi gkk'' + \xi g(k')^2 - \xi_x gkk' - a\xi k' - \xi_{xx} gk^2 - a\tau_t k - a\xi_x k = 0. \end{aligned}$$

We can distinguish the following two cases:

Case 1. $2gk' + g'k = 0$. Setting $a = 1$ we get that $k(x) = \frac{1}{x}$, and $g(x) = x^2$. The generators of the Lie algebra are

$$\begin{aligned} &\mathbf{v_1} = \partial_t, \\ &\mathbf{v_2} = \partial_v, \\ &\mathbf{v_3} = 2t\partial_t + u\partial_u + v\partial_v, \\ &\mathbf{v_4} = \frac{xv}{2}\partial_x + t\partial_v - \frac{u^2}{2}\partial_u, \\ &\mathbf{v_5} = x\partial_x, \\ &\mathbf{v_6} = \left(\frac{xv^2}{4} + \frac{tx}{2}\right)\partial_x + t^2\partial_t + tv\partial_v - \left(\frac{vu^2}{2} - tu\right)\partial_u, \\ &\mathbf{v_\beta} = x^2\beta\partial_x - (x\beta u + \beta_v xu^2)\partial_u, \end{aligned} \tag{20}$$

where $\beta(t,v)$ satisfies a linear diffusion equation $\beta_t - \beta_{vv} = 0$. We point out that $\mathbf{v_\beta}$ is an infinite-dimensional nonlocal symmetry that allows us to linearize Eq. (15).

This function $g(x) = x^2$ was already derived in Ref. [13] by deriving the classical symmetries of an integrated equation with a general weighted variable. The generators appearing in Ref. [13] are the corresponding generators of the integrated equation.

By checking (7) we can assure that $\mathbf{v_4}$ and $\mathbf{v_6}$ are potential generators while $\mathbf{v_3}$ projects onto a classical generator.

Case 2. $2gk' + g'k \neq 0$ then the corresponding determining equations give rise to

$$\xi = \frac{gk(\psi_v - \tau_t)}{2gk' + g'k}, \tag{21}$$

$$\tau = k_3 t + k_4, \tag{22}$$

$$\psi = k_1 v^2 + \frac{k_3}{2} v + k_6. \tag{23}$$

After setting $\frac{k'}{k} = f$, the compatibility of the remaining determining equations leads to

$$f'' - \frac{5}{4}\frac{(f')^2}{f} - \frac{ff'}{2} - \frac{f^3}{4} = 0. \tag{24}$$

A particular solution of (24) is $f = \frac{d}{x}$ with $d = -1$ or $d = 3$. By setting $c = a = 1$ we get $g(x) = x^2$ already considered in Case 1 and $g(x) = -\frac{1}{x^2}$. For $g(x) = -\frac{1}{x^2}$ the generators of the Lie algebra are

$$\begin{aligned}
\mathbf{w_1} &= \partial_t, \\
\mathbf{w_2} &= \partial_v, \\
\mathbf{w_3} &= 2t\partial_t + u\partial_u + v\partial_v, \\
\mathbf{w_4} &= xv\partial_x + v^2\partial_v - (x^4u^2 + 2uv)\partial_u, \\
\mathbf{w_5} &= x\partial_x + 2v\partial_v - 2u\partial_u.
\end{aligned} \tag{25}$$

We point out that for $g(x) = -\frac{1}{x^2}$ Eq. (15) admits a *new hidden potential* symmetry $\mathbf{w_4}$ which, as far as we know, has not been derived by considering the natural potential system [19] nor has been derived by considering the general weighted integral variable [13]. We have derived this generator by considering the hidden potential system

$$\begin{aligned}
v_x &= x^3 u, \\
v_t &= -\frac{ax}{3}\frac{u_x}{u^2} - \frac{a}{u}.
\end{aligned} \tag{26}$$

3. A porous medium equation

The quasi-linear equation

$$u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x \tag{27}$$

corresponds to porous media with sources or thermal evolution with sources and convection. This equation exhibits a wide variety of wave phenomena, some of them were studied for $f(x)$ and $g(x)$ *constant* by Rosenau and Kamin [20].

There is no fundamental reason to assume the spatial-dependent factors in (27) to be constant. Actually, allowing for their spatial dependence enables us to incorporate additional factors into the study which may play an important role. For instance, in a porous medium this may account for stationary factors like mediums contamination with another material or in plasma, this may express the impact that solid impurities coming from the walls, have on the enhancement of the radiation channel.

In Ref. [6], we have obtained a complete potential symmetry classification for the porous medium equation when it can be written in a conserved form

$$u_t = [(u^n)_x + f(x)u^m]_x, \tag{28}$$

through the point symmetry classification of the *natural* related potential system

$$\begin{aligned}
v_x &= u, \\
v_t &= (u^n)_x + f(x)u^m.
\end{aligned} \tag{29}$$

We found that (28) admits potential symmetries for $n = -1$ and $n = 1$ and some special values of the parameter m and some special functions $f(x)$. The nonlinear equation (27) with $n = -1$, $m = 1$ and $n = -1$, $m = -1$, does not admit an infinite-parameter Lie group of contact transformations, so cannot be linearizable by an invertible contact (point) transformation. Nevertheless the natural potential system (29) admits some infinite-parameter Lie groups, by using them we have also linearized (28) by explicit non-invertible mappings [6].

In Ref. [7], we have derived nonclassical symmetries for the porous medium equation with convection

$$u_t = (u^n)_{xx} + \frac{f(x)}{m}(u^m)_x. \tag{30}$$

We are now considering the more general auxiliary system (12) and we require the governing PDE (30) with $m = n$ to be expressed in a conserved form by this system, this yields to

$$\begin{aligned} v_x &= a(x)u \\ v_t &= a(x)nu^{n-1}u_x + cu^n, \end{aligned} \tag{31}$$

where $f(x) = \frac{n(c+a')}{a}$ and $c = \textit{constant}$. If system (31), with $n = -1$, is invariant under a Lie group of point transformations with infinitesimal generator (6) then

$$\begin{aligned} &\xi = \frac{a\psi_v}{a'} - \frac{ak_3}{2a'}, \qquad \tau = \tau(t), \qquad \psi = \psi(t, v), \\ &\phi = -a\xi_v u^2 + (\psi_v - \xi\frac{a'}{a} - \xi_x)u, \end{aligned} \tag{32}$$

and the following conditions must be satisfied

$$\begin{aligned} &2a^2\psi_{vvv} + 2\psi_{tv} - \tau_{tt} = 0, \\ &\psi_{vv}(2ca^2a' - 2a^3a'' + 3a^2(a')^2) - (a')^2\psi_t = 0, \\ &(ca'a'' + aa'a''' - 2a(a'')^2 + (a')^2a'')(2\psi_v - \tau_t) = 0. \end{aligned}$$

By solving this system we get that in order to get nonlocal symmetries

$$\begin{aligned} &\tau = k_3t + k_4, \\ &\psi = k_1v^2 + \left(\frac{k_3}{2} + k_2\right)v + (k_5t + k_6), \end{aligned} \tag{33}$$

where $a(x)$ must satisfy the following equations

$$ca'a'' + aa'a''' - 2a(a'')^2 + (a')^2a'' = 0, \tag{34}$$

$$k_1(4a^3a'' - 4ca^2a' - 6a^2(a')^2) + k_5(a')^2 = 0. \tag{35}$$

The compatibility of (34) and (35) requires that $k_5 = 0$ and the solution can be given in implicit form. A particular exact solution is $a(x) = -\frac{2cx}{3}$, $f(x) = \frac{1}{2x}$. The generators of the Lie algebra are

$$\begin{aligned} &\mathbf{v_1} = \partial_t, \\ &\mathbf{v_2} = \partial_v, \\ &\mathbf{v_3} = x\partial_x - u\partial_u + v\partial_v, \\ &\mathbf{v_4} = 2xv\partial_x + \left(\frac{4cx^2u^2}{3} - 2uv\right)\partial_x + v^2\partial_v, \\ &\mathbf{v_5} = 2t\partial_t + u\partial_u + v\partial_v. \end{aligned} \tag{36}$$

Generator $\mathbf{v_4}$ induces a nonlocal symmetry admitted by Eq. (30).

4. The Fokker–Planck equation

The classical symmetries for the Fokker–Planck equation with drift

$$u_t = u_{xx} + [f(x)u]_x, \tag{37}$$

were derived in Ref. [3]. The authors found [3] that, besides the infinite-dimensional generator, when $f(x)$ satisfies any of these Riccati equations

$$\frac{f'}{2} - \frac{f^2}{4} = ax^2 + bx + c, \tag{38}$$

$$\frac{f'}{2} - \frac{f^2}{4} = a(x+\lambda)^2 + c + \frac{d}{(x+\lambda)^2}, \tag{39}$$

then (37) is invariant under a Lie group with four or two parameters respectively.

The classical potential symmetries were derived by Pucci and Saccomandi in Ref. [16] by using the *natural* potential system

$$\begin{aligned} v_x &= u, \\ v_t &= u_x + f(x)u. \end{aligned} \tag{40}$$

They found [16] that, besides the infinite-dimensional generator, when $f(x)$ satisfies any of these Riccati equations

$$\frac{f'}{2} + \frac{f^2}{4} = ax^2 + bx + c, \tag{41}$$

$$\frac{f'}{2} + \frac{f^2}{4} = a(x+\lambda)^2 + c + \frac{d}{(x+\lambda)^2}, \tag{42}$$

then (37) is invariant under a Lie group with four or two parameters respectively.

In Ref. [15] Priestly and Clarkson found that the solutions arising from the nonclassical symmetries of the associated potential system of the shallow water equation were obtainable by the nonclassical symmetries of the shallow water equation. Consequently, it remained as an open problem the existence of nonclassical potential symmetries, in the sense that they lead to *new* solutions.

We have studied in Refs. [8,9] the nonclassical symmetries of the Fokker–Planck equation, as well as the *nonclassical potential symmetries*.

We are now considering the general auxiliary system (12) and we require the governing PDE (37) to be expressed in conserved form as this system, and system (12) becomes

$$\begin{aligned} v_x &= a(x)u, \\ v_t &= a(x)u_x + \left(\frac{a(x)a(x)''}{a(x)'} - a(x)'\right)u, \end{aligned} \tag{43}$$

with $f(x) = \frac{a''(x)}{a'(x)}$. If system (43) is invariant under a Lie group of point transformations with infinitesimal generator (6) then we get

$$\begin{aligned} &\xi = \frac{\tau'(t)x}{2} + \varphi(t), \qquad \psi = \alpha(x,t)v + \beta(x,t), \\ &\tau = \tau(t), \qquad \phi = \left(\psi_v - \xi_x - \frac{a'}{a}\xi\right)u + \frac{\xi_x}{a}, \end{aligned} \tag{44}$$

where $\alpha(x,t)$, $\beta(x,t)$, $a(x)$, $\tau(t)$ and $\varphi(t)$ must satisfy the following equations

$$(\tau'x + 2\varphi)\gamma' + \left[\tau' - \frac{a'}{a}(\tau'x + 2\varphi)\right]\gamma + \tau''x + 2\varphi' - \frac{a'}{a}\tau' + 4\alpha_x = 0,$$

$$aa'\alpha_{xx} + (aa'a'' - 2(a')^2)\alpha_x - aa'\alpha_t = 0,$$

$$aa'\beta_{xx} + (aa'a'' - 2(a')^2)\beta_x - aa'\beta_t = 0,$$

with $\gamma(x) = \frac{a(x)''}{a(x)'} - \frac{a(x)'}{a(x)}$. In general it is difficult to find all solutions of the overdetermined nonlinear system of PDEs. Next we consider the following solution

$$\xi = 1, \qquad \tau = c, \qquad \psi = \tan(x)v,$$
$$\phi = \left(\tan(x) + \frac{2(1+\cos(2x))}{2x+\sin(2x)}\right)u + \frac{k_1}{4}\sec^2(x)(2x+\sin(2x))v, \tag{45}$$

with

$$a = -\frac{4}{k_1(\sin(2x)+2x)}, \tag{46}$$

and

$$f = -\frac{2(\cos(4x) + 4x\sin(2x) + 8\cos(2x) + 7)}{\sin(4x) + 2\sin(2x) + 4x\cos(2x) + 4x}.$$

From generator (45) we obtain the similarity variable and the similarity solution

$$z = x - ct, \qquad v = \frac{h(z)}{\cos(x)}. \tag{47}$$

By introducing (47) into the generalized integrated equation

$$v_t - v_{xx} - \left(\frac{a(x)''}{a(x)'} - \frac{2a(x)'}{a(x)}\right)v_x = 0, \tag{48}$$

we get the ODE

$$h'' + ch' + h = 0. \tag{49}$$

Some solutions of (48), for $a(x)$ given by (46) are:

$$v = \frac{(k_2(x-2t)+k_3)e^{-(x-2t)}}{\cos(x)}, \tag{50}$$

$$v = \frac{e^{-\frac{c(x-ct)}{2}}}{\cos(x)}[k_2\sinh(k(x-ct)) + k_3\cosh(k(x-ct))], \tag{51}$$

for $(c-2)(c+2) > 0$, $k = \frac{1}{2}\sqrt{c^2-4}$,

$$v = \frac{e^{-\frac{c(x-ct)}{2}}}{\cos(x)}[k_2\sin(k(x-ct)) + k_3\cos(k(x-ct))], \tag{52}$$

for $(c-2)(c+2) < 0$, $k = \frac{1}{2}\sqrt{4-c^2}$.

The corresponding solutions for (37) are given by $u = \frac{v_x}{a}$. It can be checked that f *does not* satisfy any of the Riccati equations (38)–(42) consequently generator (45) cannot be derived as a Lie symmetry [3], nor as a potential symmetry by considering the natural potential system [16]. Generator (45) is a *new* potential symmetry derived from the *hidden* potential system (43).

5. Concluding remarks

In this paper we have proved that by considering a generalized potential system new potential symmetries can be derived. We have proposed as examples an inhomogeneous diffusion equation, a porous medium equation and a family of Fokker–Planck equations with drift. These *hidden* potential symmetries have not been derived in Ref. [16], for the Fokker–Planck equation by considering the *natural* potential system, nor for the inhomogeneous nonlinear equation (15) in Ref. [19], by considering the *natural* potential system nor in Ref. [13] by considering a general integral variable.

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