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Common non-trivial invariant closed cones for commuting contractions ☆

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Abstract

Let $T = (T_1, \ldots, T_N)$ be a system of N commuting contractions defined on a infinite dimensional separable Hilbert space H. In this article, we will prove that if $(1, \ldots, 1) \in \sigma_{He}(T) \bigcap \mathbb{T}^N$, where $\sigma_{He}(T)$ denotes the essential Harte spectrum of T and $\mathbb{T}^N$ the unit politorus, respectively, then there exists a non-trivial cone C invariant for each contraction T_j; $j \in \{1, \ldots, N\}$. This result complements recent results of Tsatsomeros and co-workers [Roderick Edwards, Judith J. McDonald, Michael J. Tsatsomeros, On matrices with common invariant cones with applications in neural and gene networks, Linear Algebra Appl. 398 (2005) 37–67; Michael Tsatsomeros, A criterion for the existence of common invariant subspaces of matrices, Linear Algebra Appl. 322 (1–3) (2001) 51–59].

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1. Introduction

Let $T = (T_1, \ldots, T_N)$ be a N-tuple of commuting contractions defined on a Hilbert space H. We say that $C \subset H$ is a cone of H if $rC + sC \subset C$ for each $r, s \geqslant 0$. We say that T has a non-trivial invariant closed cone if there exists a closed cone $C \subset H$, $C \neq H, \{0\}$ such that $T_j(C) \subset C$ for each $j \in \{1, \ldots, N\}$.

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Similary we say that $T = (T_1, \ldots, T_N)$ has a non-trivial invariant closed subspace if there exists $F \subset H$, $F \neq H$, $\{0\}$ closed subspace such that $T_j(F) \subset F$ for each $j \in \{1, \ldots, N\}$.

If H is finite dimensional, the existence of common invariant cones and subspaces have been studied previously in [1,6]. This problem is very interesting for its applications to neural and gene networks.

From a theoretical point of view, we study the existence of common invariant cones for commuting contractions on the infinite dimensional setting. The problem turns really complicated when H is infinite dimensional. In fact, it is not even known the existence of a non-trivial closed invariant subspace or cone for a single operator defined on a separable Hilbert space. This is know as "The invariant subspace problem".

Let T be a bounded linear operator defined on a separable Banach space X. Let us say that T is cyclic if there exists $x \in X$ such that

$$Linear\ Span\{T^n x : n \in \mathbb{N}\}$$

is dense in X. In this case x is called a cyclic vector for T. With this new terminology in hand, an operator T has a non-trivial invariant closed subspace if and only if T has a non-cyclic vector x.

On the other hand we say that T is supercyclic if there exists $x \in X$ such that $\{\lambda T^n x; \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X. In this case x is called a supercyclic vector for T. The existence of a non-supercyclic vector is in general a weaker condition than the existence of a invariant closed cone. For N-tuple of operators the existence of a non-trivial supercyclic vector and the existence of a invariant closed cone are not related. In [7] it was proved that under some hypothesis about the Harte spectrum of $T = (T_1, \ldots, T_N)$, $T_1, \ldots, T_N$ have a common non-supercyclic vector. For supercyclic operators and related questions we refer to the survey by Montes–Salas (see [3]).

The next section is devoted to prove our main result (Theorem 2.2). From our main result we being able to give a spectral sufficient condition which guarantees the existence of a common invariant closed cone for $T_1, \ldots, T_N$ and complements the results by Tsatsomeros an co-workers (see [1,6]).

2. Main result

Let T be a contraction defined on a Hilbert space H. Let us consider the subspaces

$$M_1 = \{x \in H : T^n x \to 0\}$$

and

$$M_2 = \{x \in H : (T^*)^n x \to 0\}.$$

M_1 and M_2 are invariant closed subspaces for T and T^*, respectively. To find a non-trivial invariant closed subspace is sufficient to consider the cases $M_1 = \{0\}, H$ and $M_2 = \{0\}, H$. Depending of the posibles combinations of M_1 and M_2, the classes $C_{00}, C_{10}, C_{01}, C_{11}$ are defined. For example $T \in C_{11}$ if $M_1 = M_2 = H$. Concretely in this case a result of Nagy–Foias guarantee the existence of a hyperinvariant closed subspace for T (see [5]).

In the case of an N-tuple $T = (T_1, \ldots, T_N)$, after the same reduction we have the cases $T_j \in C_{0\cdot}$ or $T_j \in C_{\cdot 0}$. We will guarantee the existence of a non-trivial invariant closed cone for the cases $T_j \in C_{0\cdot}$, $j \in \{1, \ldots, N\}$ or $T_j \in C_{\cdot 0}$, $j \in \{1, \ldots, N\}$, and under some spectral sufficient conditions.

Now we will see our main result. We will need the following technical lemma whose proof can be found in [4].

Lemma 2.1. *There exists a sequence of positive numbers* (c_i), $i \in \mathbb{N}$ *such that* $\sum_{i=1}^{\infty} c_i^2 = 1$ *and* $\sum_{i=k+1}^{\infty} c_i^2 > 3c_k$ *for all* $k \geqslant 1$.

Theorem 2.2. *Let* $T=(T_1, \ldots, T_N)$ *be a* N*-tuple of commuting contractions such that* $(1, \ldots, 1) \in \sigma_{He}(T) \cap \mathbb{T}^N$ *and* $T_j \in C_{0\cdot}$ *for each* $j \in \{1, \ldots, N\}$. *If* $\{\alpha_n\} \subset [0, 1]$ *is a sequence of real positive numbers converging to zero, then there exists* $x_0 \in H \setminus \{0\}$ *such that for all multi-index* $n = (n_1, \ldots, n_N)$ *with* $n_j \geqslant 1$ *for each* $j \in \{1, \ldots, N\}$

$$\operatorname{Re}\langle T_1^{n_1} \ldots T_N^{n_N} x_0, x_0\rangle \geqslant \alpha_{j(n)},$$

where $j(n)$ *is defined by* $j(n) = \max\{n_1, \ldots, n_N\}$.

Proof. Since $T_1, \ldots, T_N$ are contractions, the point $(1, \ldots, 1)$ is a boundary point, that is, $(1, \ldots, 1) \in \partial\sigma_{He}(T)$ and therefore, $(1, \ldots, 1) \in \sigma_{\pi e}(T)$, here $\sigma_{\pi e}(T)$ denotes the upper semi-Fredholm spectrum. Therefore, for all $\varepsilon > 0$ and $M \subset H$ of finite codimension, there exists $v \in M$ of norm 1 such that

$$\|T_j v - v\| < \varepsilon; \quad j \in \{1, \ldots, N\}.$$

We can suppose without loss of generality that the sequence (α_n) is decreasing, in other case we will replace it for the sequence $(\sup\{\alpha_j : j \geqslant n\})$. Let (c_n) be another real numbers sequence satisfying the conditions of Lemma 2.3. Let (δ_n) be a sequence of real numbers satisfying

$$\delta_i < \frac{1 - \alpha_1}{2^i}$$

and

$$\delta_i < \min\left\{\frac{c_k}{1 - 2^{i-k+1}} k \in \{1, \ldots, i+1\}\right\}.$$

Since $\alpha_i \to 0$, then there exists m_0 such that $\alpha_{m_0} < \sum_{i=2}^{\infty} c_i^2 - 3c_i$. We will construct by induction two sequences $(m_k)_{k=1}^{\infty} \subset \mathbb{N}$ and $(x_i)_{i=1}^{\infty} \subset H$ from the following form.

Let us suppose that $k \in \mathbb{N}$, and that $x_i \in H$ and $m_i \in \mathbb{N}$ have been already constructed for $i < k$. We take $M = \{v \in H : v \perp T_1^{n_1} \ldots T_N^{n_N} x_i$ for $0 \leqslant n_j \leqslant m_{k-1}$ and $i < k\}$. since M is of finite codimension, there exists x_k of norm 1 such that

$$x_k \perp T_1^{n_1} \ldots T_N^{n_N} x_i$$

for $0 \leqslant n_j \leqslant m_{k-1}$ and $i < k$ and such that

$$\|T_1^{n_1} \ldots T_N^{n_N} x_k - x_k\| \leqslant \delta_k.$$

By hypothesis $T_j \in C_{0\cdot}$. Let us consider $m_k > m_{k-1}$ large enough such that

$$\|T_j^n x_i\| < \delta_k, \quad j \in \{1, \ldots, N\} \tag{1}$$

for all $n \geqslant m_k$ and all $j \in \{1, \ldots, N\}$, and

$$\alpha_{m_k} < \sum_{i=k+2}^{\infty} c_i^2 - 3c_{k+1}.$$

However, since $T_1, \ldots, T_N$ are contractions, we deduce from (1) that

$$\|T_1^{s_1} \ldots T_N^{s_N} x_i\| \leqslant \delta_k \quad \text{for all } i \leqslant k$$

and whenever $\max\{s_1, \ldots, s_N\} \geqslant m_k$.

Let us suppose that the sequences (m_k) and (x_k) have been constructed. Let us consider the vector $x_0 = \sum_{i=1}^{\infty} c_i x_i$. Since the sequence (x_i) is orthonormal $\|x_0\| = 1$.

Given $n = (n_1, \dots, n_N)$ a multi-index such that $j(n) = \max\{n_1, \dots, n_k\} \leqslant m_0$, then we have

$$\begin{aligned}
\operatorname{Re}\langle T_1^{n_1} \dots, T_N^{n_N} x_0, x_0\rangle &= \operatorname{Re}\sum_{i=1}^{\infty} c_i \langle T_1^{n_1} \dots T_N^{n_N} x_i, x_0\rangle \\
&= \operatorname{Re}\sum_{i=1}^{\infty} c_i \left(\langle x_i, x_0\rangle - \left\langle \sum_{i=1}^{\infty} x_i - T_1^{n_1} \dots T_N^{n_N} x_i, x_0 \right\rangle \right) \\
&\geqslant \sum_{i=1}^{\infty} c_i^2 - \sum_{i=1}^{\infty} c_i \|x_i - T_1^{n_1} \dots T_N^{n_N} x_i\| \\
&\geqslant 1 - \sum_{i=1}^{\infty} c_i \delta_i > 1 - \sum_{i=1}^{\infty} \frac{1-\alpha_1}{2^i} \geqslant \alpha_{j(n)}. \qquad (2)
\end{aligned}$$

We use the inequality $\|T_1^{n_1} \dots T_N^{n_N} x_k - x_k\| \leqslant \delta_k$ in the third line of (2), and the inequality $\delta_i < \frac{1-\alpha_1}{2^i}$ in the fourth line.

Now let us suppose that $m_{k-1} < j(n) \leqslant m_k$ for $k \geqslant 1$, where $j(n) = \max\{n_1, \dots, n_N\}$ and $n = (n_1, \dots, n_N)$:

$$\begin{aligned}
\operatorname{Re}\langle T_1^{n_1} \dots T_N^{n_N} x_0, x_0\rangle &= \operatorname{Re}\sum_{i=1}^{k-1} c_i \langle T_1^{n_1} \dots T_N^{n_N} x_i, x_0\rangle \\
&\quad + \operatorname{Re} c_k \langle T_1^{n_1} \dots T_N^{n_N} x_k, x_0\rangle + \sum_{i=k+1}^{\infty} \operatorname{Re} c_i \langle T_1^{n_1} \dots T_N^{n_N} x_i, x_0\rangle \\
&\geqslant \sum_{i=1}^{k-1} c_i \delta_{k-1} - c_k + \sum_{i=k+1}^{\infty} c_i \|x_i - T_1^{n_1} \dots T_N^{n_N} x_i\| \\
&\geqslant -(k-1)\delta_{k-1} - c_k + \sum_{i=k+1}^{\infty} \delta_i \\
&\geqslant \sum_{i=k+1}^{\infty} c_i^2 - 3c_k \geqslant \alpha_{m_{k-1}} \geqslant \alpha_{j(n)}. \qquad (3)
\end{aligned}$$

We use the inequality $\alpha_{m_k} < \sum_{i=k+2}^{\infty} c_i^2 - 3c_{k+1}$in the last line of (3).

Therefore, the proof is complete. □

Using Theorem 2.2, we can obtain the following consequence which complement the results by Tsatsomeros and co-workers [1,6] in the finite dimensional setting.

Corollary 2.3. *Let $T = (T_1, \dots, T_N)$ be a N-tuple of commuting contractions, and let us suppose that $(1, \dots, 1) \in \sigma_{He}(T) \cap \mathbb{T}^N$ and $T_j \in C_0$. for each $j \in \{1, \dots, N\}$. Then there exists a closed cone C such that $T_j(C) \subset C$ for each $j \in \{1, \dots, N\}$.*

Proof. We can deduce from Theorem 2.2 the existence of $x_0 \in H \setminus \{0\}$ such that

$$\operatorname{Re}\langle T_1^{n_1} \dots T_N^{n_N} x_0, x_0\rangle > 0 \qquad (4)$$

for all $n_j \geqslant 1$, $j \in \{1, \ldots, N\}$. Let us denote by $\mathscr{P}_N$ the set of polynomials p in N variables, such that the summands of p have the form

$$cx_1^{n_1} \ldots x_N^{n_N}$$

with $n_j \geqslant 1$ for all $j \in \{1, \ldots, N\}$ and $c > 0$. Let us consider the cone

$$C = \{p(T_1, \ldots, T_N)(x_0) : p \in \mathscr{P}_N\}.$$

It is clear that C is invariant under each T_j, $j \in \{1, \ldots, N\}$ and by (4) C is non-trivial. □

Corollary 2.4. *Let $T = (T_1, \ldots, T_N)$ be a N-tuple of commuting contractions, and let us suppose that $(1, \ldots, 1) \in \sigma_{He}(T) \cap \mathbb{T}^N$ and $T_j \in C_{\cdot 0}$ for each $j \in \{1, \ldots, N\}$. Then there exists a non-trivial closed cone C such that $T_j(C) \subset C$ for each $j \in \{1, \ldots, N\}$.*

Proof. Since $T_j \in C_{\cdot 0}$, then by Theorem 2.2

$$\mathrm{Re}\langle (T_1^*)^{n_1} \ldots (T_N^*)^{n_N} x_0, x_0\rangle \geqslant \alpha_{j(n)} \Leftrightarrow Re\langle x_0, T_1^{n_1} \ldots T_N^{n_N} x_0\rangle \geqslant \alpha_{j(n)}.$$

Therefore

$$C = \{p(T_1, \ldots, T_N)(x_0) : p \in \mathscr{P}_N\}$$

is a common invariant closed cone. □

3. Concluding remarks and open problems

Let $T = (T_1, T_2)$ a 2-tuple of contractions. If we change the hypothesis $(1, 1) \in \sigma_{He}(T) \cap \mathbb{T}^2$ by $(\mu_1, \mu_2) \in \sigma_{He}(T) \cap \mathbb{T}^2$ we can only get to

$$T_j(C) \subset \mu_j C, \quad j \in \{1, 2\}.$$

It would be interesting if in this case we can find a common non-trivial invariant closed cone.

If $T_1 \in C_{0\cdot}$ and $T_2 \in C_{\cdot 0}$, with $(1, 1) \in \sigma_{He}(T_1, T_2)$ the existence of a common invariant closed cone is not known.

Let us consider the subspaces

$$M_1 = \{x \in H \text{ such that } T_1^n T_2^n x \to 0\}$$
$$M_2 = \{x \in H \text{ such that } (T_1^*)^n (T_2^*)^n x \to 0\}.$$

M_1 and M_2 are invariant closed subspaces for T_j and T_j^*; $j \in \{1, 2\}$, respectively, therefore, $M_2^{\perp}$ is invariant by T_1, T_2. Hence, we can suppose that M_1 and M_2 are trivial, that is $\{\theta\}$ or H.

Let us suppose that $M_1 = H = M_2$, then we said that $T = (T_1, T_2)$ is of class K_{11} and in this case there exists a common non-trivial invariant subspaces for T_1, T_2 (see [2]).

If $M_1 = \{0\}$ we say that T belongs to the class $K_{0\cdot}$, and if $M_2 = \{0\}$ T belongs to the class and $K_{\cdot 0}$. These cases are non-trivial. It would be interesting to investigate if Theorem 2.2 is true if $T \in K_{0\cdot}$ or $K_{\cdot 0}$.

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