BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 39(2), 173-182

About the statistical uniform convergence

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Abstract. In this work we study the concept of statistical uniform convergence. We generalize some results of uniform convergence in double sequences to the case of statistical convergence. We also prove a basic matrix theorem with statistical convergence.

Keywords: statistical convergence, double sequences, statistical uniform convergence.

Mathematical subject classification: 40A05.

1 Introduction

The concept of statistical convergence was introduced by Steinhaus [11] and by Fast [5] in 1951.

Other works about the study of statistical convergence are [6], [7] and [9]. In [8] Kolk begins the study of the applications of the statistical convergence to the Banach spaces. In [4] there are important results that relate the statistical convergence to classical properties of Banach spaces. In [2], the weakly unconditionally Cauchy series are characterized by the statistical convergence.

Let A be a set of natural numbers. Denote by $|A|$ the cardinal of A and if $n \in \mathbb{N}$ we denote $A(n) = \{i \in A : i \leq n\}$. The density of A is defined by $dt(A) = \lim_n \frac{1}{n}|A(n)|$, in case it exists.

In this work we denote by X a metric space with a metric d. Consider $(x_n)_n$ a sequence in X. $(x_n)_n$ is said to be statistically convergent to some $x \in X$, we write $st - \lim_n x_n = x$, if for each $\varepsilon > 0$, $dt\big(\{i \in \mathbb{N} : d(x_i, x) < \varepsilon\}\big) = 1$.

A sequence $(x_n)_n$ of X is said to be statistically Cauchy if for each $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists an integer $m \geq n$ such that $dt\big(\{i \in \mathbb{N} : d(x_i, x_m) < \varepsilon\}\big) = 1$.

Fridy [7] proved that a sequence $(x_n)_n$ is statistically convergent if and only if it is statistically Cauchy.

Salat [10] proved that $st - \lim_n x_n = x$ if and only if there exists $A \subset \mathbb{N}$ with $dt(A) = 1$ and $\lim_{n \in A} x_n = x$.

Received 18 May 2006.

Let $(x_{ij})_{i,j}$ be a double sequence in X. It is said that $(x_{ij})_{i,j}$ converges to x_0 (in Pringsheim's sense) if for each $\varepsilon > 0$ there exist $p, q \in \mathbb{N}$ such that $d(x_{ij}, x_0) < \varepsilon$, if $i \geq p$ and $j \geq q$. It is said that $(x_{ij})_{i,j}$ is Cauchy (in Pringsheim's sense) if for each $\varepsilon > 0$ there exist $p, q \in \mathbb{N}$ such that $d(x_{pq}, x_{ij}) < \varepsilon$, if $i \geq p$, $j \geq q$.

If X is complete we have that a double sequence $(x_{ij})_{i,j}$ is Cauchy if and only if it is convergent. Observe that a double sequence $(x_{ij})_{i,j}$ which is Cauchy is not necessarily bounded.

Let A be a subset of $\mathbb{N} \times \mathbb{N}$. It is said that the density of A is $\alpha \in [0, 1]$ if there exists the double limit

$$dt_2(A) = \lim_{p,q} \frac{|A(p,q)|}{pq} = \alpha,$$

where $A(p,q) = \{(i,j) \in A : i \leq p \, , \, j \leq q\} \, , \, (p,q) \in \mathbb{N} \times \mathbb{N}$.

It is said that the double sequence $(x_{ij})_{i,j}$ is statistically convergent to x_0 if for each $\varepsilon > 0$ it is satisfied that $dt_2\big(\{(i,j) : d(x_{ij}, x_0) < \varepsilon\}\big) = 1$. A double sequence $(x_{ij})_{i,j}$ is said to be statistically Cauchy if for each $\varepsilon > 0$ there exist $p, q \in \mathbb{N}$ such that $dt_2\big(\{(i,j) \in \mathbb{N} \times \mathbb{N} : d(x_{ij}, x_{pq}\big) < \varepsilon\}) = 1$.

Moricz, in [9], proved that if X is complete then every double sequence $(x_{ij})_{i,j}$ which is Cauchy is also convergent. He also proved that $st - \lim_{i,j}(x_{ij}) = x_0$ if and only if there exists $A \subset \mathbb{N} \times \mathbb{N}$ with $dt_2(A) = 1$ and such that $(x_{ij})_{(i,j)\in A}$ is convergent to x_0 (in Pringsheim's sense).

If we use the completion CX of the metric space X we deduce that:

i) If $(x_i)_i$ is a statistically Cauchy sequence of X then there exists a subset $A \subset \mathbb{N}$ such that $dt(A) = 1$ and $(x_i)_{i\in A}$ is Cauchy.

ii) If $(x_{ij})_{i,j}$ is a statistically Cauchy double sequence then there exists $A \subset \mathbb{N} \times \mathbb{N}$ with $dt_2(A) = 1$ and such that $(x_{ij})_{(i,j)\in A}$ is Cauchy.

In this work we introduce the following concepts:

We say that $(x_{ij})_{i,j}$ is strongly statistically convergent to x_0 and we write $Sst - \lim x_{ij} = x_0$ if there exists $K \subset \mathbb{N}$ with $dt(K) = 1$ and such that $(x_{ij})_{(i,j)\in K\times K}$ is convergent to x_0.

We say that $(x_{ij})_{i,j}$ is strongly statistically Cauchy if there exists $K \subset \mathbb{N}$ with $dt(K) = 1$ and such that $(x_{ij})_{(i,j)\in K\times K}$ is Cauchy.

This concept is more exigent than the double statistical limit of a sequence but it will allow us obtain better results related to uniform convergence.

It is clear that if $K \subset \mathbb{N}$ and $dt(K) = 1$ then $dt_2(K \times K) = 1$, so if $(x_{ij})_{i,j}$ is strongly statistically convergent (or strongly statistically Cauchy) then $(x_{ij})_{i,j}$ is statistically convergent (or statistically Cauchy).

But the converse is not true as we see in the next example:

Consider $N_1 = \{1, 3, 5, 7, \ldots\}$, $N_2 = \{1, 2, 3, 5, 6, 7, 9, \ldots\}, \ldots$, $N_k = \mathbb{N} \setminus \{m2^k : m \in \mathbb{N}\}$. We have that $dt(N_k) = 1 - \frac{1}{2^k}$ if $k \in \mathbb{N}$. Consider $A = \{(i, j) : j \in N_i\}$. We have that $dt_2(A) = 1$. Suppose that there exists $K \subset \mathbb{N}$ with $dt(K) = 1$ and $K \times K \subset A$. Fix $i \in K$, then for each $j \in K$ it will be $(i, j) \in K \times K \subset A$, so $j \in N_i$ and $K \subset N_i$, but this is a contradiction because $dt(N_i) = 1 - \frac{1}{2^i}$.

If we fix a vector x_0 in the metric space X and consider the double sequence $(x_{ij})_{i,j}$ in X where

$$x_{i,j} = \begin{cases} x_0 & \text{if } (i, j) \in A \\ 0 & \text{otherwise} \end{cases}$$

we have that $(x_{ij})_{i,j}$ is statistically convergent to x_0 but it is false that $(x_{ij})_{i,j}$ is strongly statistically convergent to x_0. It is also easy to find examples of double sequences that are statistically Cauchy whereas not strongly statistically Cauchy.

In this work we will obtain a double sequence result related to uniform convergence. We can find it partially and without proof in [1] and here we will give a simple proof of it.

Our purpose is to finish the work with a section where we will study double sequences results for the statistical convergence.

2 Uniform convergence of double sequences

Theorem 1. *Let $(x_{ij})_{i,j}$ be a double sequence in a metric space X such that $\lim_j x_{ij} = x_{i0}$, for each i and $\lim_i x_{ij} = x_{0j}$, for each j. Then the following assumptions are equivalent:*

1. $\lim_j x_{ij} = x_{i0}$, *uniformly on i.*
2. $\lim_i x_{ij} = x_{0j}$, *uniformly on j.*
3. $(x_{ij})_{i,j}$ *is Cauchy in Pringsheim's sense.*

In this situation we have that the sequences $(x_{i0})_i$ and $(x_{0j})_j$ are Cauchy and in the completion CX of X it is satisfied that $\lim_i x_{i0} = \lim_j x_{0j} = \lim_{ij} x_{ij}$, *i.e., we have that* $\lim_i \lim_j x_{ij} = \lim_j \lim_i x_{ij} = \lim_{ij} x_{ij}$.

Proof. $1 \Rightarrow 2$. Let $\varepsilon > 0$. We have that there exists j_0 such that if $p, q \geq j_0$ then $d(x_{ip}, x_{iq}) < \frac{\varepsilon}{4}$ for each i, so we deduce that $d(x_{0p}, x_{0q}) < \frac{\varepsilon}{4}$ if $p, q \geq j_0$. Fix $p > j_0$. Since $x_{ip} \underset{i\to\infty}{\longrightarrow} x_{0p}$ we have that there exists i_1 such that if $i \geq i_1$ then

$$d(x_{ip}, x_{0p}) < \frac{\varepsilon}{4}, \quad \text{so}$$
$$d(x_{ij}, x_{0j}) \leq d(x_{ij}, x_{ip}) + d(x_{ip}, x_{0p}) + d(x_{0p}, x_{0j}) \leq \varepsilon$$

if $j > j_0$ and $i \geq i_1$.

For $j \in \{1, \ldots, j_0\}$ there exists i_2 such that if $i \geq i_2$ then $d(x_{ij}, x_{0j}) < \varepsilon$, so if $i \geq i_0 = \max\{i_1, i_2\}$ it is $d(x_{ij}, x_{0j}) < \varepsilon$ for every $j \in \mathbb{N}$.

In the same manner we can see that $2 \Rightarrow 1$.

It is easy to prove that $3 \Rightarrow 1$ and we are going to see that 1 and 2 implies 3.

Let $\varepsilon > 0$. We have that there exists j_0 such that if $p, q \geq j_0$ it is $d(x_{ip}, x_{iq}) < \varepsilon/2$ for each i and there also exists i_0 such that if $p, q \geq i_0$ it is $d(x_{pj}, x_{qj}) < \varepsilon/2$ for each j.

Let $N = \max(i_0, j_0)$. If $p > N$ and $q > N$ we have that $d(x_{NN}, x_{pq}) \leq d(x_{NN}, x_{pN}) + d(x_{pN}, x_{pq}) < \varepsilon$.

In the situation of 1, 2 and 3 we will prove that $(x_{i0})_i$ is Cauchy. Let $\varepsilon > 0$. We have that there exists N such that if $p, q \geq N$ then $d(x_{NN}, x_{pq}) < \frac{\varepsilon}{2}$, so if $p, p', q, q' \geq N$ then $d(x_{pq}, x_{p'q'}) \leq d(x_{pq}, x_{NN}) + d(x_{NN}, x_{p'q'}) \leq \varepsilon$. So, if $q' \longrightarrow \infty$ we deduce that $d(x_{pq}, x_{p'0}) \leq \varepsilon$ if $p, q, p' \geq N$ and if $q \longrightarrow \infty$ we deduce that $d(x_{p0}, x_{p'0}) \leq \varepsilon$ if $p, p' \geq N$.

Let $x_0 \in CX$ be such that $\lim_i x_{i0} = x_0$. Let $\varepsilon > 0$. We now apply the same argument as before to obtain that there exists N such that if $p, q, p' \geq N$ then $d(x_{pq}, x_{p'0}) \leq \varepsilon$, so if $p' \longrightarrow \infty$ we deduce that $d(x_{pq}, x_0) < \varepsilon$ if $p, q \geq N$.

Analogously we prove that $(x_{0j})_j$ is Cauchy, so there exists $y_0 \in CX$ such that $\lim_j x_{0j} = y_0$ and in the same manner we can see that $\lim(x_{ij}) = y_0$, so $x_0 = y_0$. □

Remark 1. If X is a metric space and $(x_{ij})_{i,j}$ is a double sequence such that for each i, $(x_{ij})_{i,j}$ is Cauchy and for each j, $(x_{ij})_{i,j}$ is a Cauchy sequence, it is satisfied that the following sentences are equivalent:

i) $(x_{ij})_{i,j}$ in uniformly Cauchy on i.

ii) $(x_{ij})_{i,j}$ is uniformly Cauchy on j.

iii) $(x_{ij})_{i,j}$ is Cauchy in Pringsheim's sense.

To prove this we only need to consider the completion CX of X.

3 Uniform statistical convergence

Let $(x_{ij})_{i,j}$ be a double sequence in X. Consider $(x_{i0})_i$, a sequence in X. We say that $(x_{ij})_{i,j}$ is strongly uniformly statistical convergent (susc) to $(x_{i0})_i$ if there exists $K \subset \mathbb{N}$ with $dt(K) = 1$ such that for each $\varepsilon > 0$, $dt\big(\{j : d(x_{ij}, x_{i0}) < \varepsilon \text{ for each } i \in K\}\big) = 1$.

In [6], A. Freedman and J.J. Sember prove the following result:

Let $\{A_i : i \in I\}$ be a countable collection of subsets of $\mathbb{N}$ such that $dt(A_i) = 1$ for each $i \in I$. Then there is a set $A \subset \mathbb{N}$ such that $dt(A) = 1$ and $|A \backslash A_i| < \infty$ for all $i \in I$.

Theorem 2. *Let X be a metric space and consider $(x_{ij})_{i,j}$, a double sequence in X such that for each i, $(x_{ij})_{i,j}$, is statistical convergent and for each j, $(x_{ij})_{i,j}$, is statistical convergent. Then the following assumptions are equivalent:*

1. *For each i, $(x_{ij})_{i,j}$, is susc.*
2. *For each j, $(x_{ij})_{i,j}$, is susc.*
3. *The double sequence $(x_{ij})_{i,j}$ is strongly statistically Cauchy.*

Proof. Let us first prove that 1 implies 2. Let $K \subset \mathbb{N}$ be with $dt(K) = 1$ and such that if $\varepsilon > 0$ then $dt\big(\{j : d(x_{ij}, x_{i0}) < \varepsilon \text{ for each } i \in K\}\big) = 1$.

If $j \in \mathbb{N}$ we define $K_j = \{n \in \mathbb{N} : d(x_{in}, x_{i0}) < 1/j \text{ for each } i \in K\}$.

An analysis similar to that used by Salat [10] is the following one: Let $v_1 \in K_1$. There exists $v_2 \in K_2$ with $v_2 > v_1$ such that if $n \geq v_2$ and $n \in K_2$ then

$$\frac{|K_2(n)|}{n} \geq 1 - \frac{1}{2}, \quad \text{where} \quad K_2(n) = \big\{i \in K_2 : i \leq n\big\}.$$

We obtain by induction the sequence $v_1 < v_2 < \ldots$ such that if $n \geq v_j$ then $\dfrac{|K_j(n)|}{n} \geq 1 - \dfrac{1}{j}$.

Observe that $K_1 \supset K_2 \supset \ldots \supset K_j \supset \ldots$ and we define

$$K_0 = (1, v_1) \cup ((v_1, v_2) \cap K_1) \cup \ldots \cup ((v_j, v_{j+1}) \cap K_j) \cup \ldots$$

It follows easily that $dt(K_0) = 1$ and $\lim_{j \in K_0} x_{ij} = x_{i0}$ uniformly in $i \in K$.

For each j there exists $B_j \subset \mathbb{N}$ with $dt(B_j) = 1$ and $\lim_{i \to \infty} x_{ij} = x_{0j}$.

Applying [6] we deduce that there exists $B \subset \mathbb{N}$ with $dt(B) = 1$ and such that $|B \backslash B_j| < \infty$ if $j \in \mathbb{N}$. If $A = K \cap K_0 \cap B$ we have that $dt(A) = 1$ and for each i, $(x_{ij})_{(i,j)\in A\times A}$ is uniformly convergent to x_{i0} if $i \in A$, so for each j, $(x_{ij})_{(i,j)\in A\times A}$ is uniformly convergent to x_{0j} if $j \in A$. Also it is satisfied that $(x_{ij})_{(i,j)\in A\times A}$ is Cauchy.

Then $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are proved.

We can see that $2 \Rightarrow 1$ as we have seen that $1 \Rightarrow 2$. An easy computation shows that $3 \Rightarrow 1$. □

Remark 2. Under the same hypotheses of the last theorem we deduce that there exists $x_0 \in CX$ such that $st-\lim_i x_{i0} = st-\lim_j x_{0j} = Sst-\lim x_{ij} = x_0$, i.e., we have that $st-\lim_i st-\lim_j x_{ij} = st-\lim_j st-\lim_i x_{ij} = Sst-\lim x_{ij}$.

Definition 1. Let $(x_{ij})_{i,j}$ be a double sequence in X and $(x_{i0})_i$ a sequence. We say that $(x_{ij})_{i,j}$ is uniformly statistically convergent to $(x_{i0})_i$ if for each $\varepsilon > 0$ it is satisfied that $dt_2\big(\{(i,j) : d(x_{ij}, x_{i0}) < \varepsilon\}\big) = 1$.

If $(x_{0j})_j$ is a sequence in X we say that $(x_{ij})_{i,j}$ is uniformly statistically convergent to $(x_{0j})_j$ if for each $\varepsilon > 0$ it is satisfied that $dt_2\big(\{(i,j) : d(x_{ij}, x_{0j}) < \varepsilon\}\big) = 1$.

Theorem 3. *Let X be a metric space and consider $(x_{ij})_{i,j}$, a double sequence in X such that for each i it is $st-\lim_j x_{ij} = x_{i0}$ and for each j it is $st-\lim_i x_{ij} = x_{0j}$. Then the following assumptions are equivalent:*

1. *$(x_{ij})_{i,j}$ is uniformly statistically convergent to $(x_{i0})_i$, for each i and $(x_{i0})_i$ is statistically convergent to x_0*
2. *$(x_{ij})_{i,j}$ is uniformly statistically convergent to $(x_{0j})_j$ for each j and $(x_{0j})_j$ is statistically convergent to x_0*
3. $st - \lim_{i,j}(x_{ij}) = x_0$

Proof. We first prove that 1 implies 3. We can proceed analogously to the work of Moricz in [9]. Let $(n_r)_r$ be a sequence of natural numbers such that $2n_r \leq n_{r+1}$ if $r \in \mathbb{N}$ and $1/(pq)|\{(i,j) : i \leq p, j \leq q$ and $d(x_{ij}, x_{i0}) > 2^{-r}\}| < 1/(2^{2r})$ if $p, q \geq n_r$. Define the double sequence $(\alpha_{ij})_{i,j}$ as follows:

If $\min(i,j) < n_1$ it is $\alpha_{ij} = x_{ij}$. If p, q satisfy that $n_p \leq i < n_{p+1}, n_q \leq j < n_{q+1}$ it is

$$\alpha_{ij} = \begin{cases} x_{ij} & \text{if } d(x_{ij}, x_0) < \dfrac{1}{2^{\min(p,q)}} \\ x_{i0} & \text{if } d(x_{ij}, x_0) > \dfrac{1}{2^{\min(p,q)}} \end{cases}.$$

Consider $K = \{(i, j) : x_{ij} = \alpha_{ij}\}$. As in [9] we can prove that $dt_2(K) = 1$ and for $(x_{ij})_{(i,j)\in K}$ it is satisfied that if we consider $\varepsilon > 0$ there exists n_0 such that if $i \geq n_0$, $j \geq n_0$ and $(i, j) \in K$ then $d(x_{ij}, x_{i0}) < \varepsilon$.

We have, by hypothesis, that there exists $K' \subset \mathbb{N}$ with $dt(K') = 1$ and $\lim_{i\in K'} x_{i0} = x_0$.

Let $K_0 = \{(i, j) \in K : i \in K'\}$. It is easy to check that $dt_2(K_0) = 1$.

Finally we have that, for $\varepsilon > 0$, there exists n_0 such that if $(i, j) \in K_0$, $i \geq n_0$ and $j \geq n_0$ then $d(x_{ij}, x_{i0}) < \frac{\varepsilon}{2}$ and $d(x_{i0}, x_0) < \frac{\varepsilon}{2}$, so $d(x_{ij}, x_0) < \varepsilon$ if $i, j \geq n_0$ and $(i, j) \in K_0$.

Then $st - \lim(x_{ij}) = x_0$.

Let us prove that 3 implies 1. The equivalence between 3 and 2 would be proved analogously.

We have that there exists n_0 such that if $i, j \geq n_0$, $(i, j) \in K$ then $d(x_{ij}, x_0) < \varepsilon/2$. Consider $H = \big\{i \in \mathbb{N} : dt(\{j : (i, j) \in K\}) \neq 0\big\}$.

It is easy to check that $dt(H) = 1$ and if $K_0 = \{(i, j) \in K\ , i \in H\}$ it is satisfied that $dt_2(K_0) = 1$.

Fix $i \in H$ with $i \geq n_0$. We have that $d(x_{ij}, x_0) < \varepsilon/2$ if $j \geq n_0$ with $(i, j) \in K_0$. If $j \longrightarrow \infty$ we deduce that $d(x_{i0}, x_0) < \varepsilon/2$ if $i \geq i_0$. So, if $(i, j) \in K_0$ and $i \geq n_0$, $j \geq n_0$ it is $d(x_{ij}, x_{i0}) \leq d(x_{ij}, x_0) + d(x_{i0}, x_0) < \varepsilon$. Then $(x_{ij})_{i,j}$ is uniformly statistically convergent to $(x_{i0})_i$. □

Remark 3.

a) Observe that with the same hypotheses of the last theorem it is satisfied that $st - \lim_i(st - \lim_j x_{ij}) = st - \lim_j(st - \lim_i x_{ij}) = st - \lim_{i,j} x_{ij}$.

b) We do not know whether the last theorem remains true if in 1 we do not consider the hypothesis $(x_{i0})_i$ *is statistically convergent to* x_0 and in 2 we do not consider $(x_{0j})_j$ *is statistically convergent to* x_0.

4 The Basic Matrix Theorem for the statistical convergence

In this section we denote by X a normed space.

In [3] and [12] it is proved the well known Antosik-Swartz Basic Matrix Theorem, which states:

Let $(x_{ij})_{i,j}$ *be a double sequence in a normed space* X *such that:*

i) $\lim_i x_{ij} = x_j$ *if* $j \in \mathbb{N}$.

ii) *If* B *is an infinite subset of* $\mathbb{N}$ *then there exists an infinite subset* $C \subset B$ *such that the sequence* $\big(\sum_{j\in C} x_{ij}\big)_i$ *is Cauchy.*

Then it is satisfied that $\lim_i x_{ij} = x_j$ *uniformly in* $j \in \mathbb{N}$.

The following theorem is a version of this one but with statistical convergence.

If $\sum_i x_i$ is a series in X and C is an infinite subset of N we say that the statistical summation of $\sum_{i\in C} x_i$ is x_0, and we write $st - \sum_{i\in C} x_i = x_0$, if

$$st - \lim_n \left(\sum_{i\in C\cap\{1,\ldots,n\}} x_i \right) = x_0.$$

Theorem 4. *Let X be a normed space and consider* $(x_{ij})_{i,j}$ *a double sequence in X that satisfies:*

i) $st - \lim_j x_{ij} = 0$ *for each* i.

ii) $(x_{ij})_i$ *is a statistically Cauchy sequence for each* j.

iii) *For each infinite subset* $B \subset \mathbb{N}$ *there exists an infinite subset* $C \subset B$ *such that the sequence* $\left(st - \sum_{j\in C} x_{ij}\right)_i$ *is Cauchy.*

Then the double sequence $(x_{ij})_{i,j}$ *is strongly uniformly statistically Cauchy.*

Proof. From [6] we deduce that there exists $A \subset \mathbb{N}$ with $dt(A) = 1$ and such that $\lim_{j\in A} x_{ij} = 0$ if $i \in A$ and $(x_{i,j})_{i\in A}$ is Cauchy if $j \in A$.

If we prove that $(x_{i,j})_{i\in A}$ is uniformly Cauchy in $j \in \mathbb{N}$ it will be proved the theorem.

On the contrary there exists $\varepsilon > 0$ such that for each $i \in A$ there exists $k > i$, $k \in A$ and $j \in A$ such that $\|x_{ij} - x_{kj}\| > \varepsilon$.

In the rest of the proof the natural numbers considered belong to A.

For $i_1 = 1$ there exists $k_1 > i_1$ and j_1 such that $\|x_{i_1 j_1} - x_{k_1 j_1}\| > \varepsilon$.

On the other hand there exists $l_1 > j_1$ such that

$$\|x_{i_1 j} - x_{k_1 j}\| < \frac{\varepsilon}{3\cdot 2} \quad \text{if} \quad j \geq l_1.$$

Since $(x_{ij})_{i,j}$ is Cauchy if $j \in \{1,\ldots,l_1\}$, we have that there exists $p_1 > i_1$ such that if $p, q \geq p_1$ then $\sum_{j\in C} \|x_{pj} - x_{qj}\| < \frac{\varepsilon}{3}$ if $C \subset \{1,\ldots,l_1\} \cap A$.

For $i_2 > p_1$ there exist $k_2 > i_2$ and j_2 such that $\|x_{i_2 j_2} - x_{k_2 j_2}\| > \varepsilon$.

It is clear that $j_2 > l_1$ and there exists $l_2 > j_2$ such that

$$\|x_{i_1 j} - x_{k_1 j}\| < \frac{\varepsilon}{3\cdot 2^2} \quad \text{and} \quad \|x_{i_2 j} - x_{k_2 j}\| < \frac{\varepsilon}{3\cdot 2^2} \quad \text{if} \quad j > l_2.$$

Inductively we obtain the following sequences in A:

$$i_1 < k_1 < i_2 < k_2 < \ldots < i_r < k_r < \ldots$$
$$j_1 < l_1 < j_2 < l_2 < \ldots < j_r < l_r < \ldots$$

If $r > 1$ we have that:

i) $\sum_{j \in C} \|x_{i_r j} - x_{k_r j}\| < \frac{\varepsilon}{3}$ if $c \subset \{j_1, \ldots, j_{r-1}\} \cap A$.

ii) $\|x_{i_r j_r} - x_{k_r j_r}\| > \varepsilon$.

iii) $\|x_{i_r j_{r+h}} - x_{k_r j_{r+h}}\| < \frac{\varepsilon}{3 \cdot 2^{r+h}}$ if $h \geq 1$.

If $B = \{j_1, \ldots, j_r, \ldots\}$ there exists $C \subset B$ infinite such that the sequence $\left(st - \sum_{j \in C} x_{ij}\right)_{i \in \mathbb{N}}$ is Cauchy. So there exists n_0 such that if $r > n_0$ then

$$\left\| st - \sum_{j \in C} x_{i_r j} - st - \sum_{j \in C} x_{k_r j} \right\| < \frac{\varepsilon}{5}$$

but if $j = j_{r+h}$ then $\|x_{i_r j_{r+h}} - x_{k_r j_{r+h}}\| < \dfrac{\varepsilon}{3 \cdot 2^{r+h}}$.

Since $st - \sum_{j \in C} x_{i_r j}$ exists and $st - \sum_{j \in C} x_{k_r j}$ exists we have that $st - \sum_{j \in C} \left(x_{i_r j} - x_{k_r j}\right)$ exists too but since $\sum_{j \in C} \|x_{i_r j} - x_{k_r j}\| < \infty$ it is easy to deduce that $\sum_{j \in C} \left(x_{i_r j} - x_{k_r j}\right)$ exists and is the same as $st - \sum_{j \in C} \left(x_{i_r j} - x_{k_r j}\right)$, but if $r > n_0$ we have that

$$\begin{aligned} \left\| \sum_{j \in C} (x_{i_r j} - x_{k_r j}) \right\| &= \left\| \sum_{j \in \{j_1, \ldots, j_{r-1}\}} (x_{i_r j} - x_{k_r j}) + (x_{i_r j_r} - x_{k_r j_r}) \right. \\ &\quad \left. + \sum_{j \in \{j_{r+1}, \ldots\}} (x_{i_r j} - x_{k_r j}) \right\| \geq \varepsilon - \frac{2\varepsilon}{3} = \frac{\varepsilon}{3}, \end{aligned}$$

and this is a contradiction. □

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