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Existence of weakly efficient solutions in nonsmooth vector optimization

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Abstract

In this paper we study the existence of weakly efficient solutions for some nonsmooth and nonconvex vector optimization problems. We consider problems whose objective functions are defined between infinite and finite-dimensional Banach spaces. Our results are stated under hypotheses of generalized convexity and make use of variational-like inequalities.

Keywords: Multiobjective optimization; Nonsmooth analysis; Invexity; Variational-like inequalities

1. Introduction

The connection between variational inequalities and optimization problems is well known (e.g. [1,6,11,12]) and have been extensively investigated in the recent years by several authors. One of the main works in this direction was done by Giannessi [8], where many existence results for optimization problems were obtained by using variational inequalities.

For multiobjective optimization problems, Giannessi proved in [9] that there exists an equivalence between efficient solutions of differentiable convex optimization problems and the solutions of a variational inequality of Minty type. He also established similar results for efficient solutions. On the other hand, using subdifferentials, Lee showed in [13] that analogous results are true for nonsmooth convex problems defined between finite-dimensional spaces.

For some nonconvex differentiable vector problems defined between infinite-dimensional Banach spaces, Chen and Craven [4] proved the equivalence of weakly efficient solutions and the solutions of a certain

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variational-like inequality. Using this characterization they proved an existence result for weakly efficient solutions.

In this work we consider the following two problems.

1. An infinite-dimensional problem:

$$\begin{aligned} &\text{Minimize} \quad f(x) \\ &\text{subject to} \quad x \in K, \end{aligned} \tag{P1}$$

   where X and Y are two Banach spaces, $f : X \to Y$ is a given function and K is a nonempty subset of X.
2. A finite-dimensional problem:

$$\begin{aligned} &\text{Minimize} \quad f(x) := (f_1(x), \ldots, f_p(x)) \\ &\text{subject to} \quad x \in X, \end{aligned} \tag{P2}$$

   where $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, p)$ are given functions and X is a nonempty subset of $\mathbb{R}^n$.

For both problems, by "minimize" we mean "find the weakly efficient solution of the problem" Our objective is to solve problem (P1) without assuming hypotheses of differentiability, which extends early results by Chen and Craven [4], and to solve problem (P2) under assumptions of generalized convexity, which extends early results by Lee [13].

This paper is organized as follows: In Section 2 we fix the notation and recall some facts from nonsmooth analysis. In Section 3 we consider the problem (P1) and establish our existence result. In Section 4 we consider the problem (P2).

2. Preliminaries

Let X and Y be two real Banach spaces. We will denote by $\|\cdot\|$ the norm in Y. Let K be a nonempty subset of X and $P \subset Y$ a pointed convex cone (i.e. $P \bigcap (-P) = \{0\}$) such that $\operatorname{int} P \neq \emptyset$. Also, let $f : X \to Y$ be a given function. We consider the problem (P1) given in the previous section. The notion of optimality (or *equilibria*) that we consider here is the *weak efficiency*. We say that $x_0 \in K$ is a *weakly efficient solution* of (P1) if

$$f(x) - f(x_0) \notin -\operatorname{int} P, \quad \forall x \in K.$$

In particular, for the problem (P2), the definition of weakly efficient solution is done by taking $P = \mathbb{R}^p_+$ in the previous definition, that is, $x_0 \in X$ is a weakly efficient solution of (P2) if does not exist $x \in X$ such that

$$f_i(x) < f_i(x_0), \quad \forall i = 1, \ldots, p.$$

Now, we recall some notions and results from nonsmooth analysis. Let ϕ be a locally Lipschitz function from a Banach space X into $\mathbb{R}$. The *Clarke generalized directional derivative* of ϕ, at a point $\bar{x} \in X$, and in the direction $d \in X$, denoted by $\phi^0(\bar{x}; d)$, is given by:

$$\phi^0(\bar{x}; d) = \limsup_{\substack{y \to \bar{x} \\ t \downarrow 0}} \frac{\phi(y + td) - \phi(y)}{t}$$

and the *Clarke generalized gradient* of ϕ at $\bar{x}$ is given by

$$\partial\phi(\bar{x}) = \{x^* \in X^* : \phi^0(\bar{x}; d) \geqslant \langle x^*, d\rangle, \ \forall d \in X\},$$

where X^* denotes the topological dual of X and $\langle \cdot, \cdot \rangle$ is the canonical bilinear form pairing X^* and X.

The next proposition establish some properties of the generalized directional derivative and the generalized gradient of Clarke.

Proposition 1. *Let $f : \Omega \to \mathbb{R}$ be a locally Lipschitz function with Lipschitz constant k. Then the following assertions are true*:

1. *The function* $v \mapsto f^0(x; v)$ *is finite, sublinear and satisfies*

$$|f^0(x; v)| \leqslant k\|v\|.$$

2. *For each* $x \in \Omega, \partial f(x)$ *is a* w^**-compact and nonempty subset of* X^*. *Furthermore,* $\|\xi\| \leqslant k$, *for all* $\xi \in \partial f(x)$, *where*

$$\|\xi\| = \sup_{\substack{x \in X \\ \|x\| \leqslant 1}} \langle \xi, x \rangle, \quad \xi \in X^*.$$

3. *For each* $v \in X$, *we have*

$$f^0(x; v) = \max\{\langle \xi, v \rangle : \xi \in \partial f(x)\}.$$

4. *One has* $\xi \in \partial f(x)$ *if and only if* $f^0(x; v) \geqslant \langle \xi, v \rangle$ *for each* $v \in X$.
5. *The function* $(x; v) \mapsto f^0(x; v)$ *is upper semicontinuous.*

A function $f : \Omega \to \mathbb{R}$ is called *regular* (or *Clarke regular*) at $x \in \Omega$, if

(i) For each $v \in X$, there exists the usual directional derivative of f at x, in the direction v, denoted by $f'(x; v)$.
(ii) For all $v \in X, f'(x; v) = f^0(x; v)$.

Furthermore, if f is regular at $x \in \Omega$, for each $x \in \Omega$, then we say that f is regular in Ω. The following result holds (see [5]).

Proposition 2. *These following assertions are true*:

(a) *If* f_i *are regular at* $x \in \Omega$, *then* $\partial\left(\sum_{i=1}^n f_i\right)(x) = \sum_{i=1}^n \partial f_i(x)$. (*The inclusion* $\subset$ *is true*, *without the regularity assumption*).
(b) *If* f *is convex and Lipschitz near* $x \in \Omega$ *then* f *is regular at* x.
(c) *If* f *is continuously differentiable at* $x \in \Omega$, *then* $\partial f(x) = \{f'(x)\}$, *where* f' *is the usual derivative.*

We note that if C is a nonempty subset of X, then the *distance function* $d_C : E \to \mathbb{R}$, defined by $d_C(x) = \inf\{\|x - c\| : c \in C\}$, is not differentiable, but it is (globally) Lipschitz, with Lipschitz constant equal to 1.

Let C be a nonempty subset of X and $x \in C$. We say that $v \in X$ is a tangent vector to C at x if $d^0(x; v) = 0$. We denote by $T_C(x)$ the set of tangent vectors to C at x and $T_C(x)$ is called *tangent cone* of C at x. The *normal cone* of C at x is defined by

$$N_C(x) = \{\xi \in X^* : \langle \xi, v \rangle \leqslant 0, \ \forall v \in T_C(x)\}.$$

It can be proved that, for each $x \in C, T_C(x)$ is a convex cone, closed in X, and $N_C(x)$ is a convex cone, w^*-closed in X^*.

The following Proposition establishes a necessary condition for optimality.

Proposition 3. *Let* $f : \Omega \to \mathbb{R}$ *be a locally Lipschitz function and let* x^* *be a minimum of* f *in* $C \subset \Omega$. *Then*

$$0 \in \partial f(x^*) + N_C(x^*). \tag{1}$$

Note that the condition (1) is equivalent to

$$f^0(x^*; v) \geqslant 0, \quad \forall v \in T_C(x^*). \tag{2}$$

For more details on nonsmooth analysis, we refer the reader to the book by Clarke [5].

Next, we recall the definition of *strongly compactly* Lipschitz function, which is very important for the analysis of the infinite-dimensional problem (P1).

Definition 4 [19]. A function $h : X \to Y$ is said to be strongly compactly Lipschitz at $\bar{x} \in X$ if there exists a multifunction, that is, a point-to-set-map, $R : X \to \text{Comp}(Y)$, where $\text{Comp}(Y)$ denotes the set of all compact subsets of Y, and there exists a function $r : X \times X \to \mathbb{R}_+$ satisfying

(i) $\lim_{\substack{x \to \bar{x} \\ d \to 0}} r(x,d) = 0$.
(ii) Exists $\alpha \geqslant 0$ such that

$$t^{-1}[h(x+td) - h(x)] \in R(d) + \|d\| r(x,t) B_Y$$

for all $x \in \bar{x} + \alpha B_Y$ and $t \in (0, \alpha)$, where B_Y denotes the closed unit ball around the origin of Y.
(iii) $R(0) = \{0\}$ and R is an upper semicontinuous multifunction.

Moreover, we say that h is a strongly compactly Lipschitz function if it is strongly compactly Lipschitz at all $\bar{x} \in X$.

If Y has finite dimension, then h is strongly compactly Lipschitz at $\bar{x}$ if and only if it is locally Lipschitz near $\bar{x}$. If h is strongly compactly Lipschitz, then $(u^* \circ h)(x) = \langle u^*, h(x) \rangle$ is locally Lipschitz, for all $u^* \in Y^*$. This fact is very important because it allows us to extend some results of the nonsmooth analysis to functions defined between infinite-dimensional spaces. For more details about strongly compactly Lipschitz functions, we refer the reader to [19].

3. The infinite-dimensional problem (P1)

We recall some definitions of generalized convexity for functions defined between Banach spaces. Given a cone $P \subset Y$, the dual cone of P is defined by

$$P^* = \{\xi \in Y^* : \langle \xi, x \rangle \geqslant 0, \ \forall x \in P\}.$$

It can be proved that, if P is a convex cone, with $\text{int}\, P \neq \emptyset$, then $\langle \xi, v \rangle > 0$, for all $v \in \text{int}\, P$ (see [10]). We will need the following concepts of generalized convexity:

Definition 5 [15]. We say that a locally Lipschitz function $\theta : K \subset X \to \mathbb{R}$ is invex on K, with respect to η, if for any $x, y \in K$, there exists a vector $\eta(x,y) \in T_K(y)$ such that

$$\theta(x) - \theta(y) - \theta^0(y; \eta(x,y)) \geqslant 0.$$

Definition 6 [2]. Let X and Y be two Banach spaces and suppose that $P \subset Y$ is a convex cone. We say that $f : K \subset X \to Y$ is P-invex on K, with respect to η, if $\omega^* \circ f : K \to \mathbb{R}$ is invex on K with respect to η, for each $\omega^* \in P^*$.

Certainly, if $f : K \subset X \to \mathbb{R}$ is differentiable and convex on K, then f is invex and $\eta(x,y) = x - y$.

Note that when K is open, or more generally, if $y \in \text{int}\, K$, we have $T_K(y) = X$ and, in this case, the above definition coincides with that given by Weir and Jeyakumar [20]. Moreover, when Y is infinite-dimensional and f is differentiable, the Definition 6 is the same to that given by Santos et al. in [17]. Recall that the set K is called *invex* with respect to η if the vector $y + \alpha\eta(x,y)$ is in K for each $x, y \in K$ and $\alpha \in [0,1]$.

In the sequence, we suppose that X and Y are two Banach spaces, K is a nonempty subset of X and $P \subset Y$ is a convex cone such that $\text{int}\, P \neq \emptyset$.

We consider the following *vector variational-like inequality*.

(VI) Find $x_0 \in K$ such that, for each $x \in K$, there exists $\omega^* \in P^* \setminus \{0\}$ such that

$$(\omega^* \circ f)^0(x_0; \eta(x, x_0)) \geqslant 0.$$

Under suitable hypotheses, each solution of (VI) is a weakly efficient solution of (P1). In effect we have.

Theorem 7. *Let K be an invex set with respect to η and $f : K \subset X \to Y$ be a P-invex function with respect to η. Then each solution of (VI) is a weakly efficient solution of (P1).*

Proof. Suppose that x_0 is solution of (VI) which is not a weakly efficient solution of (P1). Then there exists $x \in K$ such that

$$f(x) - f(x_0) \in -\mathrm{int}\, P. \tag{3}$$

On the other hand, there exists $\omega^* \in P^* \setminus \{0\}$ such that $(\omega^* \circ f)^0(x_0; \eta(x, x_0)) \geqslant 0$. Since f is P-invex, it follows from (3) that

$$\omega^* \circ f(x) - \omega^* \circ f(x_0) \geqslant (\omega^* \circ f)^0(x_0, \eta(x, x_0)) \geqslant 0. \tag{4}$$

On the other hand, it follows from (3) that

$$\omega^*(f(x) - f(x_0)) = \omega^* \circ f(x) - \omega^* \circ f(x_0) < 0,$$

which contradicts (4). □

The following Lemma will be useful in the proof of our main result of this section.

Lemma 8 (KKM-Fan Theorem [7]). *Let X be a topological vector space, $E \subset X$ be a nonempty set and $F : E \rightrightarrows X$ a multifunction such that for each $x \in E$, the set $F(x)$ is closed and nonempty. Moreover, suppose that there exists some $x \in E$ such that $F(x)$ is compact. If for each finite subset $\{x_1, \ldots, x_n\}$ of E one has*

$$\mathrm{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i),$$

where $co\{x_1, \ldots, x_n\}$ is the convex hull of $\{x_1, \ldots, x_n\}$, then

$$\bigcap_{x \in E} F(x) \neq \emptyset.$$

Now we establish our existence result for (P1).

Theorem 9. *Let X be a reflexive Banach space and K a closed, convex and bounded subset of X. Let $f : K \to Y$ be a strongly compactly Lipschitz function, P-invex respect to η. Suppose that for each $x \in K$ and each $\omega^* \in P^* \setminus \{0\}$, the sets*

$$\Phi(x, \omega^*) := \{y \in K : (\omega^* \circ f)^0(x; \eta(y, x)) < 0\}$$

are convex. Furthermore, assume that η is continuous and $\eta(x, x) = 0$ for each $x \in K$. Then problem (P1) has a weakly efficient solution.

Proof. For $y \in K$ and $\omega^* \in P^* \setminus \{0\}$ we define

$$F(y, \omega^*) := \{x \in K : (\omega \circ f)^0(x; \eta(y, x)) \geqslant 0\}.$$

By Theorem 7, it is sufficient to prove that the variational-like inequality (VI) has a solution, that is,

$$\bigcap_{y \in K} \bigcup_{\omega^* \in P^* \setminus \{0\}} F(y, \omega^*) \neq \emptyset.$$

To this end, we will prove that

$$\bigcap_{y \in K} \bigcap_{\omega^* \in P^* \setminus \{0\}} F(y, \omega^*) \neq \emptyset.$$

Consider the multifunction $G : K \rightrightarrows X$, defined by

$$G(y) = \bigcap_{\omega^* \in P^* \setminus \{0\}} F(y, \omega^*).$$

We will prove that $\bigcap_{y \in K} G(y) \neq \emptyset$ by using of Lemma 8. Suppose that the space X is equipped with the weak topology. Note that $G(y)$ is nonempty for all $y \in K$. In fact, $\eta(y, y) = 0$ and thus $y \in F(y, \omega^*)$ for all $\omega^* \in P^* \setminus \{0\}$. Hence $y \in G(y)$. Furthermore, $G(y)$ is closed. In fact, let $y \in K$ and $(x_k) \subset G(y)$ be a sequence such that $x_k \to x$. Then $(\omega^* \circ f)(x_k; \eta(y, x_k)) \geqslant 0$ for all $\omega^* \in P^* \setminus \{0\}$. Fixed $\omega^* \in P^* \setminus \{0\}$, we have

$$\limsup_{k \to \infty} (\omega^* \circ f)^0(x_k; \eta(y, x_k)) \geqslant 0. \tag{5}$$

But the function $(\omega^* \circ f)^0(\cdot,\cdot)$ is upper semicontinuous (see Proposition 1). Hence, it follows from (5) that

$$(\omega^* \circ f)^0(x;\eta(y,x)) \geqslant 0,$$

that is, $x \in G(y)$ and thus $G(y)$ is closed. But $G(y)$ is convex and hence, weakly closed.

On the other hand, K is weakly compact, because it is a convex, closed and bounded subset of X and X is a reflexive space. So $G(y)$ is weakly compact.

Now, take $x_1,\dots,x_n \in K$. We should prove that $\mathrm{co}\{x_1,\dots,x_n\} \subset \bigcup_{i=1}^n G(x_i)$.

Suppose that this is false. Then there exist $\alpha_i \geqslant 0$, $i=1,\dots,n$, such that $\sum_{i=1}^n \alpha_i = 1$ and $x := \sum_{i=1}^n \alpha_i x_i \notin \bigcup_{i=1}^n G(x_i)$. Hence, for each i, there exists $\omega_i^* \in P^* \setminus \{0\}$ such that $x \notin F(x_i,\omega_i^*)$, that is, $(\omega_i^* \circ f)^0(x;\eta(x_i,x)) < 0$.

Next we construct a $\omega^* \in P^* \setminus \{0\}$ such that $x_i \in \Omega(x,\omega^*)$ for all i, which will contradict the convexity of $\Omega(x,\omega^*)$ since $\eta(x,x)=0$.

Define for $i,j \in \{1,\dots,n\}, m \in \mathbb{N}$,

$$\omega_{i,j}^{(m)} = \begin{cases} \omega_j^* & \text{if } (\omega_j^* \circ f)^0(x;\eta(x_i,x)) < 0, \\ \frac{1}{m}\omega_j^* & \text{if } (\omega_j^* \circ f)^0(x;\eta(x_i,x)) \geqslant 0, \end{cases}$$

$$\omega_i^{(m)} = \sum_{j=1}^n \omega_{i,j}^{(m)}.$$

Clearly $\omega_i^{(m)}$ are linear, and for each $m, \omega_i^{(m)}$ are not all zeros, for $i=1,\dots,n$. Furthermore, $\omega_i^{(m)}$ are continuous. In fact, define

$$J_1 = \{j : (\omega_j^* \circ f)^0(x;\eta(x_i,x)) < 0\}, \quad \alpha_1 = \#J_1,$$
$$J_2 = \{j : (\omega_j^* \circ f)^0(x;\eta(x_i,x)) \geqslant 0\}, \quad \alpha_2 = \#J_2,$$
$$M = \max_{j=1,\dots,n} \|\omega_j^*\|,$$

where $\#J$ is the number of elements of J.

Then, for all $u \in Y$, we have

$$|\omega_i^{(m)}(u)| = \left|\sum_{j\in J_1}\omega_j^*(u) + \frac{1}{m}\sum_{j\in J_2}\omega_j^*(u)\right| \leqslant \sum_{j\in J_1}\|\omega_j^*\|\|u\| + \frac{1}{m}\sum_{j\in J_2}\|\omega_j^*\|\|u\| = \left(\alpha_1 + \frac{1}{m}\alpha_2\right)M\|u\|$$

that is, $\omega_i^{(m)} \in Y^*$. Furthermore, $\omega_i^{(m)}(u) \geqslant 0$, for each $u \in P$ (because $\omega_i^{(m)}$ is a nonnegative linear combination of functionals in P^*).

On the other hand,

$$\begin{aligned}(\omega_i^{(m)} \circ f)^0(x_i;\eta(x_i,x)) &= \left[\sum_{j\in J_1}\omega_j^* \circ f + \frac{1}{m}\sum_{j\in J_2}\omega_j^* \circ f\right]^0(x,\eta(x_i,x)) \\ &\leqslant \sum_{j\in J_1}(\omega_j^* \circ f)^0(x;\eta(x_i,x)) + \frac{1}{m}\sum_{j\in J_2}(\omega_j^* \circ f)^0(x;\eta(x_i,x))\end{aligned}$$

and hence, by taking m sufficiently big, say, $m \geqslant m(i)$, we have

$$(\omega_i^{(m)} \circ f)^0(x_i;\eta(x_i,x)) \leqslant \sum_{j\in J_1}(\omega_j^* \circ f)^0(x;\eta(x_i,x)) < 0, \quad \forall m \geqslant m(i).$$

Take $M = \max_{1\leqslant i\leqslant n} m(i)$ and define $\omega^{(M)} = \sum_{i=1}^n \omega_i^{(M)}$. Then we have $\omega^{(M)}$ is in $P^* \setminus \{0\}$ and

$$(\omega^{(M)} \circ f)^0(x;\eta(x_i,x)) \leqslant \sum(\omega_i^{(M)} \circ f)^0(x;\eta(x_i,x)) < 0$$

and so $x_i \in \Phi(x;\omega^{(M)})$ for all $i=1,\dots,n$. But $\Phi(x;\omega^{(M)})$ is convex and thus $x \in \Phi(x;\omega^{(M)})$. This contradicts $\eta(x,x)=0$. Therefore, it follows from Lemma 8, that $\bigcap_{y\in K} G(y) \neq \emptyset$. The proof is complete. □

Remark 10. We notice that if K is compact and f is continuous, then (P1) has a weakly efficient solution (see Sawaragi et al. [18]). Thus, if we apply Theorem 9 to the problem (P2), we do not obtain any new result, because the hypotheses considered there imply that K is compact and f is continuous (see Brézis [3]).

4. The finite-dimensional problem (P2)

In this Section, we will consider a variational-like inequality of (weak) Minty type and we establish a characterization of weakly efficiency for (P2), in terms of the solutions of that inequality. Using this result and a fixed point theorem for multifunctions, we establish our existence result for (P2).

Our variational-like inequality is:

($WMVLI$) Find $y \in X$ such that, for each $x \in X$ and each $\xi_i \in \partial f_i(x), i = 1, \ldots, p$,

$$(\xi_1^{\mathrm{T}}\eta(x,y), \ldots, \xi_p^{\mathrm{T}}\eta(x,y)) \notin -\mathrm{int}\,\mathbb{R}_+^p.$$

In [16], Santos et al. proved that, under certain hypotheses, the solutions of ($WMVLI$) are weakly efficient solutions of (P2).

Proposition 11 [16]. *Let X be a nonempty set of $\mathbb{R}^n$, invex with respect to η, and $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, p$ are invex functions, locally Lipschitz with respect to η. Suppose that η is skew (that is, $\eta(x,y) = -\eta(y,x), \forall x, y \in X$). Then $y \in X$ is a weakly efficient solution of (P2) if and only if $y \in X$ is a solution of ($WMVLI$).*

Now, we recall a fixed-point theorem of Fan–Browder, whose proof can be found in Park [14].

Lemma 12 [14]. *Let X be a nonempty convex subset of a Hausdorff topological vector space E and let K be a nonempty compact subset of X. Suppose that $A, B : X \rightrightarrows X$ are multifunctions satisfying the following conditions:*

1. *$Ax \subset Bx$ for all $x \in X$,*
2. *Bx is a convex set for all $x \in X$,*
3. *$Ax \neq \emptyset$ for all $x \in K$,*
4. *$A^{-1}y = \{x \in X; y \in Ax\}$ is an open set for each $y \in X$,*
5. *for each finite subset N of X, there exists a compact, convex and nonempty subset L_N of X such that $L_N \supset N$ and $Ax \bigcap L_N \neq \emptyset$ for all $x \in L_N \setminus K$.*

Then there is a $\bar{x} \in B\bar{x}$ (that is, $\bar{x}$ is a fixed point of B).

Next, we will use Proposition 11 and Lemma 12 to establish our result about the existence of weakly efficient solution for nonsmooth invex vectorial problem, under a weaker compactness hypothesis on the feasible set X.

Theorem 13. *Let X be a nonempty and invex subset of $\mathbb{R}^n$ with respect to η, and $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, p$, invex and locally Lipschtz functions with respect to the same η. Assume that η is skew and such that $\eta(\cdot, y)$ is convex and continuous, for each $y \in X$. Moreover, suppose that there exists a compact, convex and nonempty set $L_N \subset X$, such that $L_N \supset N$ and for all $x \in L_N \setminus K$, there is $z \in L_N$ such that there exist $\xi_i \in \partial f_i(z), i = 1, \ldots, p$ satisfying*

$$(\xi_1^{\mathrm{T}}\eta(z,x), \ldots, \xi_p^{\mathrm{T}}\eta(z,x)) \in -\mathrm{int}\,\mathbb{R}_+^p.$$

Then problem (P2) has a weakly efficient solution.

Proof. For sake of readability, we give a proof using a concise notation. We denote by $\partial f(x)$ the set $\partial f_1(x) \times \ldots \partial f_p(x), x \in X$. Let $s = (s_1, \ldots, s_p)$, where $s_i \in \mathbb{R}^n, i = 1, \ldots, p$. We denote by $s^{\mathrm{T}}\eta(x,y)$ the vector

$$(s_1^{\mathrm{T}}\eta(x,y), \ldots, s_p^{\mathrm{T}}\eta(x,y)) \in \mathbb{R}^p.$$

Let $A, B : X \rightrightarrows X$ be multifunctions defined by:

$$Ax := \{z \in X : \exists t \in \partial f(z), t^{\mathrm{T}}\eta(z,x) \in -\mathrm{int}\,\mathbb{R}_+^p\};$$

$$Bx := \{z \in X : \forall t \in \partial f(x), t^{\mathrm{T}}\eta(z,x) \in -\mathrm{int}\,\mathbb{R}_+^p\}.$$

We will prove (by using of Lemma 12) that exists $y \in K$ such that $Ay = \emptyset$, that is, y is solution of (WMVLI). By Proposition 11, this is sufficient to prove our result.

First we prove that the multifunctions A and B satisfy the conditions (1), (2), (4) and (5) of Lemma 12 and that B does not have a fixed point. So, Lemma 12 will imply the existence of $y \in K$ such that $Ay = \emptyset$.

We show that the condition (1) of Lemma 12 holds. Take $x \in X$ and $z \in Ax$. Then, exists $t = (\xi_1, \dots, \xi_p) \in \partial f(z)$ such that

$$(\xi_1^{\mathrm{T}}\eta(z,x), \dots, \xi_p^{\mathrm{T}}(z,x)) \in -\mathrm{int}\,\mathbb{R}_+^p. \tag{6}$$

Let $s = (\widehat{\xi}_1, \dots, \widehat{\xi}_p) \in \partial f(x)$. By using of the invexity of functions f_i and the skewness of η, we have that, for each $i = 1, \dots, p$,

$$\widehat{\xi}_i\eta(z,x) \leqslant f_i^0(x, \eta(z,x)) \leqslant f_i(z) - f_i(x) = -(f_i(x) - f_i(z)) \leqslant -\xi_i^{\mathrm{T}}\eta(x,z) = \xi_i^{\mathrm{T}}\eta(z,x). \tag{7}$$

Follows from (6) and (7),

$$(\widehat{\xi}_1^{\mathrm{T}}\eta(z,x), \dots, \widehat{\xi}_p^{\mathrm{T}}(z,x)) \in -\mathrm{int}\,\mathbb{R}_+^p,$$

and so $z \in Bx$.

Now, we will see that the second condition of Lemma 12 holds: Let $x \in X, z_1, z_2 \in Bx$ and $\lambda \in [0,1]$. Then, for each $s = (\xi_1, \dots, \xi_p) \in \partial f(x)$, we have

$$(\xi_1^{\mathrm{T}}\eta(z_1,x), \dots, \xi_p^{\mathrm{T}}\eta(z_1,x)), (\xi_1^{\mathrm{T}}\eta(z_2,x), \dots, \xi_p^{\mathrm{T}}\eta(z_2,x)) \in -\mathrm{int}\,\mathbb{R}_+^p. \tag{8}$$

For each $j = 1, \dots, p$, we consider $\xi_j = (\xi_j^{(1)}, \dots, \xi_j^{(n)})$, where $\xi_j^{(k)} \in \mathbb{R}, \eta(x,y) = (\eta_1(x,y), \dots, \eta_n(x,y))$, $\eta_k(x,y) \in \mathbb{R}$. Then, from the convexity of η_k and (8), we obtain

$$\begin{aligned}\xi_j^{\mathrm{T}}\eta(\lambda z_1 + (1-\lambda)z_2, x) &= \sum_{k=1}^{n} \xi_j^{(k)}\eta_k(\lambda z_1 + (1-\lambda)z_2, x) \leqslant \sum_{k=1}^{n} \xi_j^{(k)}[\lambda\eta_k(z_1,x) + (1-\lambda)\eta_k(z_2,x)] \\ &= \lambda\xi_j^{\mathrm{T}}\eta(z_1,x) + (1-\lambda)\xi_j^{\mathrm{T}}\eta(z_2,x) < 0,\end{aligned}$$

for each $j = 1, \dots, p$. Hence $\lambda z_1 + (1-\lambda)z_2 \in Bx$.

The fourth condition is proved as follows. For all $z \in X$, we will show that the set $(A^{-1}z)^c$ is closed. To do this, consider a sequence $(x_n) \subset (A^{-1}z)^c$ such that x_n converges to x. Then $x_n \notin A^{-1}z, \forall n \in \mathbb{N}$. Let $t = (\xi_1, \dots, \xi_p) \in \partial f(z)$ be such that

$$(\xi_1^{\mathrm{T}}\eta(z,x_n), \dots, \xi_p^{\mathrm{T}}\eta(z,x_n)) \notin -\mathrm{int}\,\mathbb{R}_+^p. \tag{9}$$

Because $\eta(\cdot,z)$ is continuous and skew, we have that $\eta(z,\cdot)$ is also continuous and skew and, since $(-\mathrm{int}\,\mathbb{R}_+^p)^c$ is a closed set, by taking $n \to \infty$ in (9) we obtain

$$(\xi_1^{\mathrm{T}}\eta(z,x), \dots, \xi_p^{\mathrm{T}}\eta(z,x)) \notin -\mathrm{int}\,\mathbb{R}_+^p,$$

and thus $x \in (A^{-1}z)^c$.

From the hypotheses, condition (5) of Lemma 12 also holds.

However, B does not have a fixed point, because otherwise it would exists some $x \in X$ such that $s \in \partial f(x)$ and $s^{\mathrm{T}}\eta(x,x) = 0 \in -\mathrm{int}\,\mathbb{R}_+^p$, which is an absurd. Thus, from Lemma 12, it follows that exists $y \in K$ such that $Ay = \emptyset$. □

We have the following consequence of Theorem 13.

Corollary 14. *Let X be a nonempty subset of $\mathbb{R}^n$, invex with respect to η, and suppose that η is skew and such that $\eta(\cdot,y)$ is convex and continuous. Moreover, assume that*

$$K = \{x \in X : (f_1^0(z_0; \eta(z_0,x)), \dots, f_p^0(z_0; \eta(z_0,x))) \notin -\mathrm{int}\,\mathbb{R}_+^p\},$$

is compact for some $z_0 \in X$. Then (P2) has a weakly efficient solution.

Proof. Let N be a nonempty and finite subset of X. Define $L_N = \overline{\text{co}}(N \bigcup K)$ (where $\overline{\text{co}}$ denotes the closed convex hull of A). Then, for each $x \in L_N \setminus K$ we have

$$(f_1^0(z_0;\eta(z_0,x)),\dots,f_p^0(z_0;\eta(z_0,x))) \in -\text{int}\,\mathbb{R}_+^p. \tag{10}$$

Take $z = z_0 \in K \subset L_N$ and $\xi_i \in \partial f_i(z)$. We have

$$\xi_i^T\eta(z_0,x) \leqslant f_i^0(z_0;\eta(z_0,x)), \quad i = 1,\dots,p, \tag{11}$$

and from (10) and (11), we obtain

$$(\xi_1^T\eta(z,x),\dots,\xi_p^T\eta(z,x)) \in -\text{int}\,\mathbb{R}_+^p.$$

Thus, by Theorem 13, (P2) has a weakly efficient solution. □

5. Conclusions

In this paper, we obtain an existence theorem for weakly efficient solutions of vector optimization problems defined between infinite dimensional Banach spaces whose objective function is invex and strongly compactly Lipschitz. We characterize the solutions of this problem in terms of the solutions of a variational-like inequality and, by applying this characterization and the KKM-Fan Theorem, we establish our result. The approach that we used here is similar to the one employed by Chen and Craven [4], where the authors considered a differentiable problem.

Also, we consider the vector problem defined between finite-dimensional spaces, and for this problem we also obtain a result on the existence of weakly efficient solution. Our approach is analogous to the one used by Lee [13], where it was considered the non-differentiable convex case. We use a characterization of the weakly efficiency in terms of the solutions of the (weak) Minty type inequalities and a fixed point theorem for multifunctions to prove our result.

The results that we present in this paper generalize those obtained by Chen and Craven [4] and Lee [13].

References

[1] C. Baiocchi, A. Capelo, Variational and Quasivariational Inequalities: Applications to the Free Boundary Problems, John Wiley and Sons Inc., New York, 1984.

[2] A.J.V. Brandão, M.A. Rojas-Medar, G.N. Silva, Optimality conditions for Pareto nonsmooth nonconvex programming in Banach spaces, J. Optim. Theory Appl. 103 (1) (1999) 65–73.

[3] H. Brézis, Analyse fonctionnelle: Théorie et Applications, Masson, Paris, 1983.

[4] G.Y. Chen, B.D. Craven, Existence and continuity of solutions for vector optimization, J. Optim. Theory Appl. 81 (1994) 459–468.

[5] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley-Interscience, 1983.

[6] G. Duvault, J.L. Lions, Les inéquations en Mécanique et en Physique, Dunod, Paris, 1972.

[7] K. Fan, A generalization of Tichonoff's fixed point theorem, Mathematische Annalen 142 (1961) 305–310.

[8] F. Giannessi, Theorems of the alternative, quadratic problems, and complementarity problems, in: R.W. Cottle, F. Giannessi, J.L. Lions (Eds.), Variational Inequalities and Complementarity Problems (Proceedings of the International School, Erice, 1978), John Wiley, Chichester, England, 1980, pp. 151–186.

[9] F. Giannessi, On Minty variational principle, in: New Trends in Mathematical Programming, Kluwer, 1997, pp. 93–99.

[10] I.V. Girsanov, Lectures on Mathematical Theory of Extremum Problems, Lecture Notes in Economics and Mathematical Systems, vol. 67, Springer-Verlag, 1972.

[11] R. Glowinsky, J.L. Lions, R. Trémoliéres, Numerical Analysis of Variational Inequalities, North-Holland, 1976.

[12] P.T. Harker, J.-S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications, Math. Program. 48 (1990) 161–220.

[13] G.M. Lee, On relations between vector variational inequality and vector optimization problem, in: X. Yang et al. (Eds.), Progress in Optimization, Kluwer, 2000.

[14] S. Park, Some coincidence theorems on acyclic multifunctions and applications to KKM Theory, in: Fixed Point Theory and Applications, World Scientific Publishing, River Edge, NJ, 1992, pp. 248–277.

[15] T.D. Phuong, P.H. Sach, N.D. Yen, Strict lower semicontinuity of the level sets and invexity of a locally Lipschitz function, J. Optim. Theory Appl. 87 (1995) 579–594.

[16] L.B. Santos, M. Rojas-Medar, A. Rufián-Lizana, Some relations between variational-like inequalities and efficient solutions of certain nonsmooth optimization problems, Int. J. Math. Sci. 2006 (2006), 16 (Article ID 26808), doi:10.1155/IJMMS/2006/26808.

[17] L.B. Santos, R. Osuna-Gómez, M.A. Rojas-Medar, A. Rufián-Lizana, Preinvex functions and weak efficient solutions for some vectorial optimization problems in Banach spaces, Comput. Math. Appl. 48 (2004) 885–895.
[18] Y. Sawaragi, H. Nakayama, T. Tanino, Theory of Multiobjective Optimization, Academic Press, New York, 1985.
[19] L. Thibault, On generalized differentials and subdifferentials of Lipschitz vector valued functions, Nonlinear Anal. 6 (1980) 1037–1053.
[20] T. Weir, V. Jeyakumar, A class of nonconvex functions and mathematical programming, Bull. Austr. Math. Soc. 38 (1988) 177–189.