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On Lawrence semigroups

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ABSTRACT

Lawrence semigroups arise as a tool to compute Graver bases of semigroup ideals. It is known that the minimal free resolution of semigroup ideals is characterized by the reduced homologies of certain simplicial complexes.

In this paper we study the minimal degrees of a Lawrence semigroup ideal and its first syzygy given a combinatorial characterization of the nonvanishing cycles in their associated reduced homologies. We specialize the results that appeared in [Briales, E., Campillo, A., Marijuán, C., Pisón, P., 1998. Minimal systems of generators for ideals of semigroups. J. Pure Appl. Algebra, 127, 7–30] and [Pisón-Casares, P., Vigneron-Tenorio, A., 2001. First syzygies of toric varieties and diophantine equations in congruence. Comm. Alg. 29 (4), 1445–1466] to the Lawrence semigroups.

0. Introduction

The Lawrence semigroups appear in the context of Graver basis computation of ideals. The Graver basis of an ideal is a generating set whose elements are the *primitive* binomials: $X^{\alpha} - X^{\beta}$ is primitive if there exists no binomial $X^{\alpha'} - X^{\beta'}$ in the ideal such that $X^{\alpha'}$ divides X^{α} and $X^{\beta'}$ divides X^{β} (Sturmfels, 1995). Lawrence lifting is a tool to compute Graver basis.

Besides, Lawrence semigroups and ideals are crucial in the computation of the universal Gröbner basis of semigroup ideal (Sturmfels, 1995). The Graver basis of a semigroup ideal is related to the integer programming and the test sets (Sturmfels and Thomas, 1997), and it can be computed by using the reduced Gröbner basis of its Lawrence lifting (Sturmfels, 1995, Chapter 7). To be precise, it is enough to substitute some variables by 1 in the elements of this Gröbner basis.

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In Section 1, we show briefly the equivalence between semigroups and lattices, their ideals and their Lawrence lifting.

It is easy to proof that any Lawrence semigroup, S, satisfies the property $S \cap (-S) = \{0\}$. Then there exists a minimal free resolution associated to its S-graded algebra. In this case, it is known that the degrees of the minimal generator set for the i-syzygy module are characterized by means of the i-reduced homology of some simplicial complexes Δ_m. Let $\tilde{H}_i(\Delta_m)$ be that i-reduced homology (see Briales et al. (1998a) for details). The minimal generators of degree m of the i-syzygy are in bijection with the base elements of the vector space $\tilde{H}_i(\Delta_m)$.

In Section 2, we study the simplicial complexes associated to the minimal generator set of Lawrence ideals. We characterize the connected components of these simplicial complexes (Proposition 1). This characterizes the minimal generator set of the ideal obtaining a similar result to the one which appears in Sturmfels et al. (1995, Theorem 5.3).

In Section 3, we study the first syzygies of Lawrence ideals describing the possible cycles in $\tilde{H}_1(\Delta_m)$. The F-cavities of the simplicial complex are the key. We describe these F-cavities and their cardinals. Notice that the nonvanishing cycles in $\tilde{H}_1(\Delta_m)$ are determined by some F-cavities. In Bayer et al. (2001) the authors construct the minimal free resolution of unimodular Lawrence ideals.

From the results of this paper one can obtain combinatorial algorithms to compute mathematical objects. The interest of these algorithms does not rely on their efficiency, but on the combinatorial description of these objects that they provide. In fact, the computation using Gröbner Basis Theory is more efficient, but this is only a computing method and does not shed any light on the combinatorial structure of these objects.

1. Lawrence lifting of semigroups and Lawrence lifting of lattice

Given $S \subset \mathbf{Z}^n \oplus \mathbf{Z}/a_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/a_h\mathbf{Z}$ a finitely generated semigroup with zero element, and $\{n_1, \ldots, n_r\} \subset S$ a set of generators for S, one can define its *Lawrence lifting* generalizing the Lawrence lifting of a matrix that appears in Sturmfels (1995, Chapter 7). The Lawrence lifting of S is a new semigroup S' generated by $(n_1, e_1), \ldots, (n_r, e_r), (0, e_1), \ldots, (0, e_r)$ in $\mathbf{Z}^n \oplus \mathbf{Z}/a_1 \oplus \cdots \oplus \mathbf{Z}/a_h \oplus \mathbf{Z}^r$ with $\{e_1, \ldots, e_r\}$ the standard coordinate vectors in $\mathbf{Q}^r$. Any semigroup like S' is called a *Lawrence semigroup*.

Fixing k as a commutative field, one can consider $k[X] = k[x_1, \ldots, x_r]$, the polynomial ring in r indeterminates where the S-degree of x_i is equal to n_i. We denote by X^α, where $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{N}^r$, the monomial $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$. Let $\gg$ be the natural partial order on $\mathbf{N}^r$.

It is easy to see that a Lawrence semigroup S' satisfies the hypothesis of Theorem 7.1 in Sturmfels (1995) using the lattice $\mathcal{L} = \{(\delta_1, \ldots, \delta_r) \in \mathbf{Z}^r \mid \sum_{i=1}^r \delta_i n_i = 0\} \subset \mathbf{Z}^r$ associated to S. It is well known that the ideal associated to S, I, and to $\mathcal{L}$, $I_\mathcal{L}$, are the same (Vigneron-Tenorio, 1999). They are as follows (see Herzog (1970) and Sturmfels et al. (1995)):

$$I := \left\langle X^\alpha - X^\beta | \sum_{i=1}^r \alpha_i n_i = \sum_{i=1}^r \beta_i n_i,\ \alpha_i, \beta_i \geq 0 \right\rangle = I_\mathcal{L} := \left\langle X^{\delta^+} - X^{\delta^-} | \delta \in \mathcal{L} \right\rangle,$$

where $\delta_i^+ = \max\{\delta_i, 0\}$ and $\delta^- = (-\delta)^+$.

The Lawrence lifting of $\mathcal{L}$ is a new lattice $\hat{\mathcal{L}} = \{(\delta, -\delta) \in \mathbf{Z}^{2r} \mid \delta \in \mathcal{L}\}$. This object was first defined in Sturmfels et al. (1995, Section 5). Notice that $I_{S'} = I_{\hat{\mathcal{L}}}$. Proposition 5.1 in Sturmfels et al. (1995) characterizes the Graver basis of $I_\mathcal{L}$ as the unique reduced Gröbner basis of $I_{\hat{\mathcal{L}}}$. This characterizes the Graver basis of I_S as the unique reduced Gröbner basis of $I_{S'}$.

2. Combinatoric results over Lawrence ideals

From now on, we fix the Lawrence semigroup

$$S = \langle n'_1, \ldots, n'_r, n'_{r+1}, \ldots, n'_{2r} \rangle \subset \mathbf{Z}^n \oplus \mathbf{Z}/a_1 \oplus \cdots \oplus \mathbf{Z}/a_h \oplus \mathbf{Z}^r$$

where $n'_i = (n_i, e_i)$ for all $i = 1, \ldots, r$, and $n'_i = (0, e_{i-r})$, $\forall i = r+1, \ldots, 2r$. Thus, $\Lambda = \{1, \ldots, r, r+1, \ldots, 2r\}$. Note that $S \cap (-S) = \{0\}$. One can consider the simplicial complex $\Delta_m =$

$\{F \subset \Lambda |\, m - n'_F \in S\}$, where $n'_F = \sum_{i\in F} n'_i$. This simplicial complex was first defined in Campillo and Marijuán (1991). Observe that, if F is a maximal face in Δ_m, then there is a monomial of degree m with support F.

We define the function $sym : \Lambda \to \Lambda$ by $sym(i) = \begin{cases} i+r, & i \le r \\ i-r, & i > r \end{cases}$, for $i \in \Lambda$. Let $A \subset \Lambda$, set $sym(A) := \{sym(i) |\, i \in A\}$. In the following lemma we prove an important property of the simplicial complexes Δ_m associated to a Lawrence semigroup.

Lemma 1. *If $\{i\} \in \Delta_m$ and $m = \sum_{j\in G} \gamma_j n'_j$, where $i \notin G \subset \Lambda$, then $sym(i) \in G$.*

Proof. $\{i\} \in \Delta_m$ implies that $m - n'_i \in S$. Thus, if $n'_i = (*, e_l)$ then the $(n+h+l)$th coordinate of m is nonzero. Since $i \notin G$ and $m = \sum_{j\in G} \gamma_j n'_j$, it is clear that $sym(i) \in G$. □

Proposition 1. *Let Δ_m be nonconnected. Then:*

(1) *$\Delta_m = C \sqcup sym(C)$, where C and $sym(C)$ are the only two connected components of Δ_m.*
(2) *$1 \le \natural C \le r$.*
(3) *C and $sym(C)$ are full subcomplexes.*

Proof. (1) Let A, B be two different connected components of Δ_m, then $A \cap B = \emptyset$ and $m = \sum_{i\in A} \alpha_i n'_i = \sum_{i\in B} \beta_i n'_i$. Using Lemma 1 one obtains $sym(A) \subset B$ and $sym(B) \subset A$. As $sym()$ is an idempotent function, the equalities hold.
(2) We know that $C \sqcup sym(C) \subseteq \Lambda$. If $\natural C > r$ then $\natural \Lambda > 2r$, but this is not possible.
(3) Suppose that C is not a full subcomplex. In that case there exist $A, B \subset C$ maximal faces of C, such that $A \neq B$ and $\exists \alpha_i, \beta_i \in \mathbf{N} \setminus \{0\}$, $m = \sum_{i\in A} \alpha_i n'_i = \sum_{i\in B} \beta_i n'_i$.
Let $i \in A$ and $i \notin B$. Then $sym(i) \in B$. We have $i, sym(i) \in C$, but this is not possible. □

The above proposition can be obtained from Sturmfels et al. (1995, Proposition 5.1). When that result is specialized to the lattice $\hat{\mathcal{L}}$ we get Proposition 1, namely, the minimal generator fibers of the associated ideals contain exactly 2 monomials which are symmetric; $X_C - X_{sym(C)}$ where X_C and $X_{sym(C)}$ are the unique monomials of degree m with support C and $sym(C)$. This can be used to obtain a combinatorial algorithm to compute the minimal generating set of I, like Briales et al. (1998b, Algorithm 5.1).

3. Combinatoric results over first syzygies of Lawrence ideals

In this section, we are going to make a combinatorial study of the first syzygies of the Lawrence semigroup ideal I. First of all, we introduce the concept of F-cavity (see Pisón-casares and Vigneron-Tenorio (2001) for details).

Definition 1. Let $m \in S$ and $F = \{i_1, \ldots, i_t\} \subset \Lambda$ such that $\natural F \ge 3$, and let σ be a polygon whose vertex set is F. We say σ is an F-cavity of Δ_m if the following conditions are satisfied:

(1) $F_j \in \Delta_m$, $\forall j = 1, \ldots, t$ where $F_j = \{i_j, i_{j+1}\}$, $\forall j = 1, \ldots, t-1$, and $F_t = \{i_t, i_1\}$, are the faces of σ.
(2) If $\forall j = 1, \ldots, t$, $F_j \neq F' \subset F$, $\natural F' \ge 2$, then $F' \notin \Delta_m$.

The relation between the F-cavities and the degrees of the first syzygies is the following (see Pisón-casares and Vigneron-Tenorio (2001, Lemma 13)).

Lemma 2. *Let $m \in S$ such that $\tilde{H}_1(\Delta_m) \neq 0$. Then, there is σ an F-cavity of Δ_m with faces F_i satisfying*

$$c = \sum_{j=1}^{t} \epsilon_j F_j \in \tilde{H}_1(\Delta_m) \setminus \{0\}, \quad \textit{for some } \epsilon_j = \pm 1, \forall j = 1, \ldots, t.$$

The vector space $\tilde{H}_1(\Delta_m)$ is the first reduced homology of Δ_m. The nonvanishing cycles of this homology, $c = \sum \epsilon_j F_j \in \tilde{H}_1(\Delta_m) \setminus \{0\}$, are isomorphic to the minimal generators of degree m of the first syzygy of I. This result appears in Briales et al. (1998a).

The particular nature of the Lawrence semigroups allows us to prove the following result.

Theorem 1. *Under the hypothesis of Lemma 2, $3 \le \natural F \le 4$. Moreover, σ has one of the shapes in Fig. 4.*

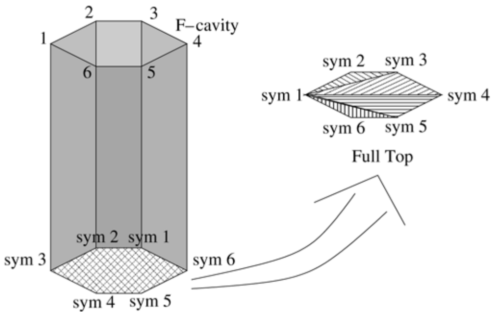

Fig. 1. $\natural F > 5$.

Proof. Let $\sigma = \{F_1, \ldots F_t\}$ be an F-cavity of Δ_m as in Definition 1, then there is $\alpha^{(i)} = (\alpha_1^{(i)}, \ldots, \alpha_{2r-t+2}^{(i)}) \in \mathbf{N}^{2r-t+2}$ satisfying

$$m = n'_{F_1} + \sum_{j \in G_1} \alpha_j^{(1)} n'_j = n'_{F_2} + \sum_{j \in G_2} \alpha_j^{(2)} n'_j = \cdots = n'_{F_{t-1}} + \sum_{j \in G_{t-1}} \alpha_j^{(t-1)} n'_j$$
$$= n'_{F_t} + \sum_{j \in G_t} \alpha_j^{(t)} n'_j,$$

where $G_l := (\Lambda \setminus F) \cup F_l$ for $l = 1, \ldots, t$. By Lemma 1, $i \in F \setminus F_l$ implies $sym(i) \in G_l$. Therefore, $\cup_{i \in F \setminus F_l} sym(i) \subset G_l$ for all $l = 1, \ldots, t$, and in particular, $\cup_{F_j \cap F_l = \emptyset} sym(F_j) \subset G_l$.

First, we are going to prove that F-cavities where $\natural F \geq 5$ do not contain both a vertex and its symmetric one. Suppose, for example, that $i_1, sym(i_1) \in F$ for some i_1. Since $\natural F \geq 5$, there is an l, such that $i_1, sym(i_1) \notin F_l$. Then, one can write m without using $n'_{i_1}, n'_{sym(i_1)}$. But this is impossible.

- Suppose that $\natural F > 5$. Notice that the following sets are in Δ_m :

$$\underbrace{sym(i_1) \cup sym(F_2),\ sym(i_1) \cup sym(F_3),\ \ldots,\ sym(i_1) \cup sym(F_{t-1})}_{\text{Full Top (base)}},$$
$$\underbrace{F_1 \cup sym(F_3),\ F_2 \cup sym(F_4),\ \ldots,\ F_{t-1} \cup sym(F_1),\ F_t \cup sym(F_2)}_{\text{Full Sides}}$$

  Thus, Δ_m *contains* a prism with an empty cover $\{i_1, \ldots, i_t\}$, a *full base* and any full sides (equivalent to Fig. 1).[3] The topological invariance of the simplicial homology groups yields $c = 0$ as an element in $\tilde{H}_1(\Delta_m)$. This is in direct conflict with Lemma 2. Therefore, $\natural F \leq 5$.
- Suppose that $\natural F = 5$. We have just seen that the unique possibility for an F-cavity with $\natural F = 5$ is the one that is in Fig. 4.

  On the other hand, notice that the following sets are in Δ_m :

$$F_1 \cup \{sym(i_3),\ sym(i_5)\},\ F_5 \cup \{sym(i_3),\ sym(i_2)\},$$
$$F_3 \cup \{sym(i_2),\ sym(i_5)\},\ F_5 \cup \{sym(i_2)\},\ F_2 \cup \{sym(i_5)\}.$$

  Thus, Δ_m *contains* a prism with an empty cover, $\{i_1, \ldots, i_5\}$, an empty base, $\{sym(i_2), sym(i_3), sym(i_5)\}$, and full sides (equivalent to Fig. 2).[4] Then all possible nonvanishing elements $c = \sum_{j=1}^{5} \epsilon_j F_j$ are equivalent to some $c' = \sum_{j=1}^{3} \epsilon'_j F'_j$ in $\tilde{H}_1(\Delta_m)$. So one can study the F-cavities with $\natural F = 3$ instead of the F-cavities with $\natural F = 5$.
- Suppose that $\natural F = 4$. Notice that there cannot be an edge like $\{i, sym(i)\}$.

[3] We have supposed that $F = \{1, 2, 3, 4, 5, 6\}$.
[4] We have supposed that $F = \{1, 2, 3, 4, 5\}$.

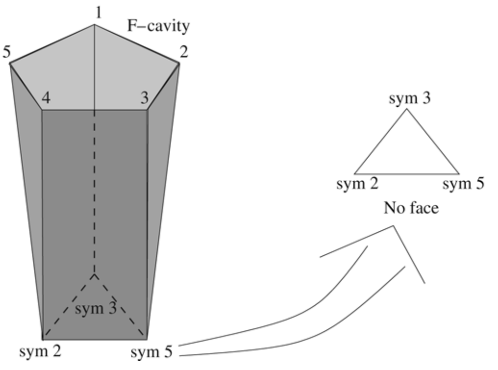

Fig. 2. $\natural F = 5$.

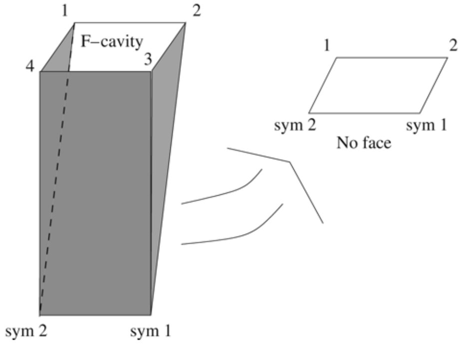

Fig. 3. $\natural F = 4$.

If one considers the F-cavity, $F = \{i_1, i_2, i_3, i_4\}$, where there are no $u, v = 1, 2, 3, 4$ such that $i_u = sym(i_v)$, the sets

$$F_4 \cup sym(i_2),\ F_2 \cup sym(i_1) \text{ and } F_3 \cup sym(F_1)$$

are in Δ_m.

Thus, Δ_m *contains* a wedge with an empty cover, $\{i_1, i_2, i_3, i_4\}$, an empty base, $\{i_1, i_2, sym(i_1), sym(i_2)\}$, and the above three full sides (equivalent to Fig. 3).[5]

Then any possible nonvanishing element $c = \epsilon_1 F_1 + \epsilon_2 F_2 + \epsilon_3 F_3 + \epsilon_4 F_4$ is equivalent to $c' = \epsilon'_1 F_1 + \epsilon'_2 \{i_2, sym(i_1)\} + \epsilon'_3 sym(F_1) + \epsilon'_4 \{i_1, sym(i_2)\}$ in $\tilde{H}_1(\Delta_m)$. So the above two F-cavities are equivalent.

The other possibility is the F-cavity $F = \{i_1, i_2, sym(i_1), i_4\}$. This possibility can happen as one can see in the complex of degree $(6, 2, 1, 3, 3)$ ($F = \{4, 5, 1 = sym(4), 6\}$) in Fig. 5.

- Suppose that $\natural F = 3$. In that case, there are only two possible F-cavities:

(1) three vertices such that no one is the symmetric of any other;
(2) three vertices such that one of them is the symmetric of another one.

These two possibilities can happen as one can see in the complex of degree $(6, 2, 1, 3, 3)$ ($F = \{1, 2, 3\}$) and $(2, 1, 3, 1, 1)$ ($F = \{1, 2, 4 = sym(1)\}$) in Fig. 5.

Then all possible F-cavities are included in Fig. 4. □

Knowing the possible cycles $c = \sum_{j=1}^{t} \epsilon_j F_j \in \tilde{H}_1(\Delta_m) \setminus \{0\}$, one can study the minimal generators for the first syzygy using the isomorphism explicited in Briales et al. (1998a, Remark 3.6).

[5] We have supposed that $F = \{1, 2, 3, 4\}$.

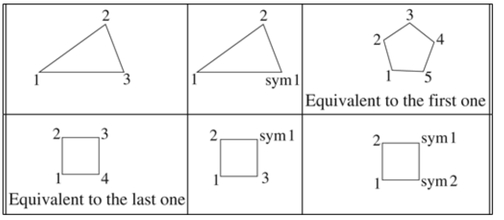

Fig. 4. Possible F-cavities.

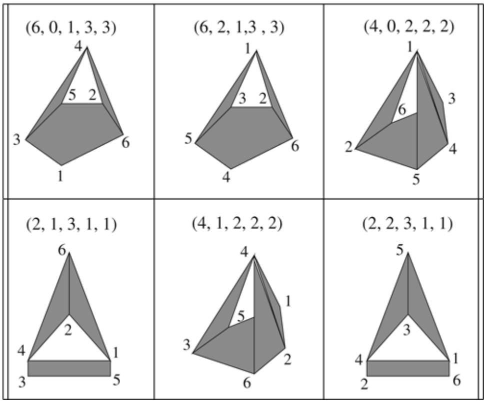

Fig. 5. m-degrees.

The Algorithm 19 to compute the first syzygy and the bound of their degrees (Theorem 23) that appear in Pisón-casares and Vigneron-Tenorio (2001) can be improved immediately for Lawrence ideals. The reason is that one must only check the possible F-cavities which satisfy that $\natural F$ is equal to 3 or 4.

We are going to illustrate Theorem 1 with an example.

Example 1. Let $S \subset \mathbf{Z} \oplus \mathbf{Z}/3 \oplus \mathbf{Z}^3$ be the Lawrence semigroup generated by

$$\langle(0,2,1,0,0),(2,1,0,1,0),(2,2,0,0,1),(0,0,1,0,0),(0,0,0,1,0),\\ (0,0,0,0,1)\rangle$$

and consider the minimal system of generators for I : $gen_1 = x_2^3x_6^3 - x_3^3x_5^3$, $gen_2 = x_1x_3x_5 - x_2x_4x_6$, $gen_3 = x_1x_2^2x_6^2 - x_3^2x_4x_5^2$, $gen_4 = x_1^2x_2x_6 - x_3x_4^2x_5$, and $gen_5 = x_1^3 - x_4^3$. A minimal system of generators for the first syzygies is

$$x_2^2x_6^2gen_2 - x_3x_5gen_3 + x_4gen_1,\ x_1gen_1 - x_2x_6gen_3 + x_3^2x_5^2gen_2,\\ x_1gen_3 - x_2x_6gen_4 + x_3x_4x_5gen_2,\ x_1gen_4 - x_2x_6gen_5 + x_4^2gen_2,\\ x_1x_2x_6gen_2 - x_3x_5gen_4 + x_4gen_3,\ x_1^2gen_2 - x_3x_5gen_5 + x_4gen_4$$

The σ F-cavities associated to their degrees are in Fig. 5.

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