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On split Lie algebras with symmetric root systems

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Abstract. We develop techniques of connections of roots for split Lie algebras with symmetric root systems. We show that any of such algebras L is of the form $L = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the abelian Lie algebra H and any I_j a well described ideal of L, satisfying $[I_j, I_k] = 0$ if $j \neq k$. Under certain conditions, the simplicity of L is characterized and it is shown that L is the direct sum of the family of its minimal ideals, each one being a simple split Lie algebra with a symmetric root system and having all its nonzero roots connected.

Keywords. Infinite dimensional Lie algebras; split Lie algebras; roots.

1. Introduction and previous definitions

Throughout this paper, Lie algebras L are considered of arbitrary dimension and over an arbitrary field $\mathbb{K}$. It is worth to mention that, unless otherwise stated, there is not any restriction on $\dim L_\alpha$, the products $[L_\alpha, L_{-\alpha}]$, or $\{k \in \mathbb{K}\colon k\alpha \in \Lambda$, for a fixed $\alpha \in \Lambda\}$.

In the study of different classes of split Lie algebras has appeared several notions of 'connections of roots' that have had an important role in the classification of the algebras under consideration. These notions are related to the particular characteristics of each class. Let us see some examples:

An L^*-algebra is defined (see [2–4]) as a complex involutive Hilbert–Lie algebra for which the inner product $(\cdot|\cdot)$ satisfies the H^*-identities

$$([x, y]|z) = (y|[x^*, z]) = (x|[z, y^*]).$$

See also [1] for a definition in the real case.

Schue proved in [4] the existence of a dense split Lie algebra, with respect to a Cartan subalgebra H, for any separable semisimple L^*-algebra, and in [3] the following notion of 'connections of roots' in order to classify them: Given two nonzero roots α, β, a chain from α to β is a finite set of nonzero roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ such that $(h_{\alpha_i}|h_{\alpha_{i+1}}) \neq 0$ for $i = 1, \ldots, n-1$, where any h_{α_i} is the unique autoadjoint element in H with $\|h_{\alpha_i}\| = 1$ and $\alpha(h) = (h|h_{\alpha_i})$ for every $h \in H$. This notion depends on the Hilbert space structure of an L^*-algebra. In the study of semisimple locally finite split Lie algebras over a field of characteristic zero $\mathbb{K}$ [5], Stumme introduces the next definition (Definition III.21 of [5]): A subset M of nonzero roots is called irreducible if for every two roots $\alpha, \beta \in M$ there exists a family of roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ such that $\alpha_j(\check{\alpha}_{j+1}) \neq 0$ for $j = 1, \ldots, n-1$, where $\check{\alpha}_{j+1}$ is the coroot associated to α_{j+1} (it is said that α and β are 'connected'). This concept holds on the existence of the coroot for any nonzero root. This fact is ensured for