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The classification question for Leavitt path algebras

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Abstract

We prove an algebraic version of the Gauge-Invariant Uniqueness Theorem, a result which gives information about the injectivity of certain homomorphisms between $\mathbb{Z}$-graded algebras. As our main application of this theorem, we obtain isomorphisms between the Leavitt path algebras of specified graphs. From these isomorphisms we are able to achieve two ends. First, we show that the K_0 groups of various sets of purely infinite simple Leavitt path algebras, together with the position of the identity element in K_0, classify the algebras in these sets up to isomorphism. Second, we show that the isomorphism between matrix rings over the classical Leavitt algebras, established previously using number-theoretic methods, can be reobtained via appropriate isomorphisms between Leavitt path algebras.

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Introduction

Throughout this article E will denote a finite directed graph, and K will denote an arbitrary field. The *Leavitt path algebra of E with coefficients in K*, denoted $L_K(E)$, has received significant attention over the past few years, both from algebraists as well as from analysts working in operator theory. (The precise definition of $L_K(E)$ is given below.) When K is the field $\mathbb{C}$ of complex numbers, the algebra $L_K(E)$ has exhibited surprising similarity to its C^*-algebra counterpart $C^*(E)$, the *Cuntz–Krieger graph C^*-algebra of E*. In part motivated by the Gauge-Invariant Uniqueness Theorem of operator theory, we prove in Section 1 of this article a result which guarantees the injectivity of certain algebra homomorphisms based on a specified action of the field as automorphisms on the codomain. With this result in hand, we use it (and other results about homomorphisms from Leavitt path algebras established here) in Section 2 to produce isomorphisms between Leavitt path algebras. Specifically, we show in Theorems 2.3 and 2.8 how, starting with the graph E and specified configurations of vertices and edges in E, to explicitly construct graphs E' having $L_K(E) \cong L_K(E')$. In Section 3 we apply these isomorphisms to obtain Proposition 3.4, a result about isomorphisms between purely infinite simple unital Leavitt path algebras. Subsequently, in Section 4 we use these isomorphisms to answer specific cases of The Classification Question for purely infinite simple unital Leavitt path algebras. We establish Propositions 4.1 and 4.2, the algebraic counterparts of specific pieces of the Kirchberg–Phillips Theorem of C^*-algebras (see [14, Section 3] for a description). These two results establish, respectively, that if E and F are finite directed graphs having two (respectively three) vertices and no parallel edges, and the Leavitt path algebras $L_K(E)$ and $L_K(F)$ are purely infinite simple, then $L_K(E) \cong L_K(F)$ if and only if the Grothendieck groups $K_0(L_K(E))$ and $K_0(L_K(F))$ are isomorphic via an isomorphism which takes $[1_{L_K(E)}]$ to $[1_{L_K(F)}]$. We close the article by showing how [4, Theorem 4.14] may be reestablished using the results and techniques of this article.

We briefly recall some graph-theoretic definitions and properties; more complete explanations and descriptions can be found in [1]. A *graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. (Some authors use the phrase 'directed' graph for this structure.) The elements of E^0 are called *vertices* and the elements of E^1 *edges*. (We emphasize that loops and multiple/parallel edges are allowed.) If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. A vertex v for which $s^{-1}(v)$ is empty is called a *sink*, while a vertex w for which $r^{-1}(w)$ is empty is called a *source*. A *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ is the *source* of μ, $r(\mu) := r(e_n)$ is the *range* of μ, and n is the *length* of μ. An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. If μ is a path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at v*. If $\mu = e_1 \dots e_n$ is a closed path based at $v = s(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. We say that a graph E satisfies *Condition* (L) if every cycle in E has an exit. For $n \geqslant 2$ we define E^n to be the set of paths of length n, and $E^* = \bigcup_{n \geqslant 0} E^n$ the set of all paths.

The following notation is standard. Let A be a $p \times p$ matrix having nonnegative integer entries (i.e., $A = (a_{ij}) \in \mathrm{M}_p(\mathbb{Z}^+)$). The graph E_A is defined by setting $(E_A)^0 = \{v_1, v_2, \dots, v_p\}$, and defining $(E_A)^1$ by inserting exactly a_{ij} edges in E_A having source vertex v_i and range vertex v_j. Conversely, if E is a finite graph with vertices $\{v_1, v_2, \dots, v_p\}$, then we define the *incidence* *matrix* A_E *of* E by setting $(A_E)_{ij}$ as the number of edges in E having source vertex v_i and range vertex v_j.

Definition 0.1. Let E be any row-finite graph, and K any field. The *Leavitt path* K*-algebra* $L_K(E)$ *of* E *with coefficients in* K is the K-algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

(1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e \in E^1 \mid s(e)=v\}} ee^*$ for every vertex $v \in E^0$ for which $s^{-1}(v)$ is nonempty.

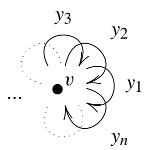
When the role of the coefficient field K is not central to the discussion, we will often denote $L_K(E)$ simply by $L(E)$.

The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \ldots e_n$ is a path, then we denote by μ^* the element $e_n^* \cdots e_1^*$ of $L_K(E)$.

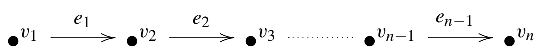
For any subset H of E^0, we will denote by $I(H)$ the ideal of $L_K(E)$ generated by H.

An alternate description of $L_K(E)$ is given in [1], where it is described in terms of a free associative algebra modulo the appropriate relations indicated in Definition 0.1 above. As a consequence, if A is any K-algebra which contains elements satisfying these same relations, then there is a (unique) K-algebra homomorphism from $L_K(E)$ to A mapping the generators of $L_K(E)$ to their appropriate counterparts in A. We will refer to this conclusion as the Universal Homomorphism Property of $L_K(E)$. See also [15, Remark 2.5].

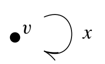
Many well-known algebras arise as the Leavitt path algebra of a row-finite graph. For instance, the classical Leavitt algebras L_n for $n \geqslant 2$ (see Definition 3.7 below) arise as the algebras $L(R_n)$ where R_n is the "rose with n petals" graph



The full $n \times n$ matrix algebra over K arises as the Leavitt path algebra of the oriented n-line graph



while the Laurent polynomial algebra $K[x, x^{-1}]$ arises as the Leavitt path algebra of the "one vertex, one loop" graph



Constructions such as direct sums and the formation of matrix rings produce additional examples of Leavitt path algebras.

If E is a finite graph then $L_K(E)$ is unital, with $\sum_{v\in E^0} v = 1_{L_K(E)}$. Conversely, if $L_K(E)$ is unital, then E^0 is finite. If E^0 is infinite then $L_K(E)$ is a ring with a set of local units; one such set of local units consists of sums of distinct elements of E^0. $L_K(E) = \bigoplus_{n\in\mathbb{Z}} L_K(E)_n$ is a $\mathbb{Z}$-graded K-algebra, spanned as a K-vector space by $\{pq^* \mid p, q \text{ are paths in } E\}$. In particular, for each $n \in \mathbb{Z}$, the *degree n* component $L_K(E)_n$ is spanned by elements of the form $\{pq^* \mid \text{length}(p) - \text{length}(q) = n\}$. The degree of an element x, denoted $\deg(x)$, is the minimum integer n for which $x \in \bigoplus_{m\leqslant n} L_K(E)_m$. The set of *homogeneous elements* is $\bigcup_{n\in\mathbb{Z}} L_K(E)_n$, and an element of $L_K(E)_n$ is said to be *n-homogeneous* or *homogeneous of degree n*. The K-linear extension of the assignment $pq^* \mapsto qp^*$ (for p, q paths in E) yields an involution on $L_K(E)$, which we denote simply as *.

Information regarding the "C^*-algebra of a graph," also known as the "Cuntz–Krieger graph C^*-algebra," may be found in [13].

1. Injectivity of algebra maps

Our central theme in this article is a description of isomorphisms between Leavitt path algebras. As we shall see, oftentimes we encounter a situation in which we have defined a surjective ring homomorphism between two such algebras, and seek to determine whether the map is injective. The main result of this section, Theorem 1.8 (which we refer to as the Algebraic Gauge-Invariant Uniqueness Theorem or AGIUT for short), provides a tool for doing just that. The AGIUT is a consequence of results for general $\mathbb{Z}$-graded K-algebras.

In fact, there are many results aside from the AGIUT which provide similar tools by which we can establish the injectivity of various algebra homomorphisms. We present two such results in the following lemmas.

Lemma 1.1. *Let E be a row-finite graph, let A be a $\mathbb{Z}$-graded K-algebra, and let $f : L_K(E) \to A$ be an algebra map such that $f(v) \neq 0$ for every $v \in E^0$. If f is graded, then f is injective.*

Proof. Since f is a graded map, $\text{Ker}(f)$ is a graded ideal of $L_K(E)$. By [6, Theorem 5.3], there exists a subset X of E^0 such that $\text{Ker}(f) = I(X)$. Since $f(v) \neq 0$ for every $v \in E^0$, we get $X = \emptyset$, whence $\text{Ker}(f) = 0$ as desired. □

The method used in the previous proof is to guarantee that $\text{Ker}(f)$ does not contain any vertices. The proof of the following lemma uses a similar line of reasoning.

Lemma 1.2. *Let E be a row-finite graph satisfying Condition* (L)*, let A be a K-algebra, and let $f : L_K(E) \to A$ be an algebra map. If $f(v) \neq 0$ for every $v \in E^0$, then f is injective.*

Proof. By [1, Lemma 3.9], Condition (L) yields that $J \cap E^0 \neq \emptyset$ for every nonzero ideal J of $L_K(E)$. Since $f(v) \neq 0$ for every $v \in E^0$, we conclude that $\text{Ker}(f) = 0$ as desired. □

With the hypotheses of the previous two lemmas in mind, we seek an injectivity result in situations in which the map is not graded, and the graph contains cycles with no exits. Such a result is the essence of the AGIUT (Theorem 1.8).

Definitions 1.3. Let K be a field, and let A be a $\mathbb{Z}$-graded algebra over K. For $t \in K^* = K \setminus \{0\}$ and a any homogeneous element of A of degree d, set

$$\tau_t(a) = t^d a,$$

and extend τ_t to all of A by linearity. It is easy to show that τ_t is a K-algebra automorphism of A for each $t \in K^*$. Then $\tau : K^* \to \mathrm{Aut}_K(A)$ is an action of K on A, which we call the *gauge action* of K on A.

If I is an ideal of A, we say that I is *gauge-invariant* in case $\tau_t(I) = I$ for each $t \in K^*$. This condition is equivalent to requiring that $\tau_t(I) \subseteq I$ for every $t \in K^*$, since $\tau_{t^{-1}}(I) \subseteq I$ gives $I \subseteq \tau_t(I)$.

The previous definition of the gauge action draws its motivation as follows. Let A be a $\mathbb{Z}$-graded K-algebra (e.g., a Leavitt path algebra $L_K(E)$) which is generated by homogeneous elements of degree 1 and -1. Then the multiplicative group K^* acts naturally on A by sending elements a of degree ϵ, where $\epsilon = 1, -1$ to $t^\epsilon a$ for each nonzero element $t \in K$. In particular if A is an involutorial K-algebra over a field K with involution, then any homogeneous element of degree -1 is the image of such an element of degree 1, and hence the unitary group of K (i.e., the group of $t \in K$ with $tt^* = 1$) acts naturally on A. This natural action of the unit circle on the Cuntz algebra $\mathcal{O}_n$ is in part the motivation for our description of an algebraic gauge action of K on A. (For additional information see [12, p. 198].)

The next result establishes a relationship between graded and gauge-invariant ideals of any $\mathbb{Z}$-graded algebra.

Proposition 1.4. *Let K be a field, let A be a $\mathbb{Z}$-graded K-algebra, and let I be an ideal of A. Let $\tau : K^* \to \mathrm{Aut}_K(A)$ be the gauge action of K on A.*

(1) *If I is generated as an ideal of A by elements of degree* 0*, then I is gauge-invariant.*
(2) *If K is infinite, and if I is gauge-invariant, then I is graded.*

Proof. Statement (1) is clear, as τ_t fixes the degree zero elements of A for each $t \in K^*$.

For statement (2), we prove the contrapositive. So suppose I is not graded. We seek to show that I is not gauge-invariant. For each $a \in A$ let $h(a)$ denote the number of nonzero homogeneous graded components of a. Since I is not graded there exists an element $a \in I$ for which, in the decomposition $a = \sum a_j$ into a sum of its homogeneous components, at least two of the a_j are not in I. Let $T \subseteq I$ denote those elements of I which, when written in homogeneous decomposition, have the property that no nonzero homogeneous component is in I. Since I is not graded, $T \neq \emptyset$. Let $b \in T$ such that $h(b) = \min\{h(t) \mid t \in T\}$. Note that $h(b) \geqslant 2$. Let $m, n \in \mathbb{Z}$ for which the homogeneous components b_m and b_n are each nonzero; assume without loss of generality that $n < m$. Because K is infinite, we can find $t \in K^*$ such that $t^m \neq t^n$. (Otherwise, we would have $t^m = t^n$ for all $t \in K^*$, so that every element of K^* would be a zero of $x^{m-n} - 1 \in K[x]$, but such cannot happen in an infinite field.)

To show that I is not gauge-invariant, it suffices to show $\tau_t(I) \nsubseteq I$. By contradiction, assume $\tau_t(I) \subseteq I$, so that in particular $\tau_t(b) \in I$. We observe that $b \in I$ gives $t^m b \in I$, so $\tau_t(b) - t^m b \in I$; we denote $\tau_t(b) - t^m b$ by c. Note that for each $i \in \mathbb{Z}$, the i-component of c is $c_i = (t^i - t^m)b_i$. Thus we have $c_m = 0$, but $c_n = (t^n - t^m)b_n \neq 0$ (and so in particular $c \neq 0$). But $c_m = 0$ gives $h(c) < h(b)$, so, by minimality, at least one of the nonzero components of c is in I. That is, for some $p \in \mathbb{Z}$, $(t^p - t^m)b_p$ is a nonzero element of I. But then b_p is a nonzero element of I, which contradicts our choice of b. □

We now apply this result in the context of Leavitt path algebras. For clarity, we present here the definition of the gauge action of K on the Leavitt path algebra $L_K(E)$ of the row-finite graph E.

Definition 1.5. Let E be a row-finite graph, and let K be a field. Then the *gauge action* τ *of* K *on* the Leavitt path algebra $L_K(E)$ (denoted sometimes by τ^E for clarity) is given by

$$\begin{aligned} \tau^E : K^* &\to \operatorname{Aut}_K\big(L_K(E)\big), \\ t &\mapsto \tau_t^E \end{aligned}$$

as follows: for every $t \in K^*$, for every $v \in E^0$, and for every $e \in E^1$

$$\begin{aligned} \tau_t^E : L_K(E) &\to L_K(E), \\ v &\mapsto v, \\ e &\mapsto te, \\ e^* &\mapsto t^{-1}e^* \end{aligned}$$

and then extend linearly and multiplicatively to all of $L_K(E)$.

For a graph E, the set of graded ideals of $A = L_K(E)$ is denoted by $\mathcal{L}_{\mathrm{gr}}$.

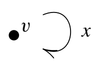
Proposition 1.6. *Let E be a row-finite graph, let K be an infinite field, and let I be an ideal of $L_K(E)$. Then $I \in \mathcal{L}_{\mathrm{gr}}$ if and only if I is gauge-invariant.*

Proof. If $I \in \mathcal{L}_{\mathrm{gr}}$, then $I = I(H)$ for some $H \subseteq E^0$ by [6, Theorem 5.3]. Thus I is generated by elements of degree zero, and so Proposition 1.4(1) applies. The converse follows immediately from Proposition 1.4(2). □

We note that the implication $I \in \mathcal{L}_{\mathrm{gr}}$ implies I is gauge-invariant holds for any field K, finite or infinite. In contrast, we now show that the converse implication of Proposition 1.6 is never true for any finite field.

Proposition 1.7. *For any finite field K there exists a graph E such that the Leavitt path algebra $L_K(E)$ contains a nongraded ideal which is gauge-invariant.*

Proof. If we denote card(K) by $m + 1$, then $t^m = 1$ for all $t \in K^*$. Let E be the graph



so that, as noted previously, $L_K(E) \cong K[x, x^{-1}]$. In particular we have $\tau_t(1 + x^m) = 1 + x^m$ for all $t \in K^*$. This then yields that the ideal $I = \langle 1 + x^m \rangle$ of $L_K(E)$ is gauge-invariant. But it is well known (or it can be shown using an argument similar to that given in the proof of [1, Theorem 3.11]) that I is not a graded ideal of $K[x, x^{-1}]$. □

We are now in position to present the main application of these ideas.

Theorem 1.8 *(The Algebraic Gauge-Invariant Uniqueness Theorem). Let E be a row-finite graph, let K be an infinite field, and let A be a K-algebra. Suppose*

$$\phi : L_K(E) \to A$$

is a K-algebra homomorphism such that $\phi(v) \neq 0$ for every $v \in E^0$. If there exists a group action $\sigma : K^* \to \mathrm{Aut}_K(A)$ such that $\phi \circ \tau_t^E = \sigma_t \circ \phi$ for every $t \in K^*$, then ϕ is injective.

Proof. Let $I = \mathrm{Ker}(\phi)$. Then for every $a \in I$ and for every $t \in K^*$, $\phi(\tau_t^E(a)) = \sigma_t(\phi(a)) = \sigma_t(0) = 0$, whence $\tau_t^E(a) \in \mathrm{Ker}(\phi) = I$. Thus for every $t \in K^*$ we have $\tau_t^E(I) \subseteq I$, so that I is gauge-invariant. Hence $I \in \mathcal{L}_{\mathrm{gr}}$ by Proposition 1.6. In particular, if $I \neq \{0\}$, then $I \cap E^0 \neq \emptyset$ by [6, Proposition 5.2 and Theorem 5.3], contradicting the hypothesis that $\phi(v) \neq 0$ for every $v \in E^0$. □

In both [3] and [15] an analysis of Leavitt path algebras for nonrow-finite graphs is carried out. We conclude Section 1 by noting that all the results (and their proofs) presented in this section hold verbatim in this more general not-necessarily-row-finite setting. In particular, Lemma 1.1 generalizes as [15, Theorem 4.8], while Lemma 1.2 generalizes as [15, Theorem 6.8].

2. Isomorphisms: general results

In this section we will apply the results of Section 1 to draw conclusions about isomorphisms between Leavitt path algebras. The main goal in establishing such isomorphisms is as follows: starting with a graph E, we seek a systematic method to produce various graphs F for which $L_K(E) \cong L_K(F)$. As such, we refer to our two main results (Theorems 2.3 and 2.8) as "Change the Graph" Theorems. These results in turn will allow us to verify that a specific set of Leavitt path algebras is determined up to isomorphism by K_0 data.

In our first such result, we show how to "bundle" specific sets of edges, and subsequently replace the bundled sets by a single edge.

Definition 2.1. Let E be a row-finite graph, and let $v \neq w \in E^0$ be vertices which are not sinks. If there exists an injective map

$$\theta : s^{-1}(w) \to s^{-1}(v)$$

such that $r(e) = r(\theta(e))$ for every $e \in s^{-1}(w)$, we define the *shift graph* from v to w, denoted

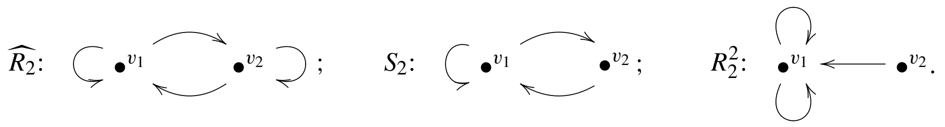
$$F = E(w \hookrightarrow v),$$

as follows:

(1) $F^0 = E^0$.
(2) $F^1 = (E^1 \setminus \theta(s^{-1}(w))) \cup \{f_{v,w}\}$, where $f_{v,w} \notin E^1$, $s(f_{v,w}) = v$ and $r(f_{v,w}) = w$.

Although the definition the graph $F = E(w \hookrightarrow v)$ depends on the map θ, in order to make the notation less cumbersome we suppress θ in the notation. This will cause no confusion throughout the sequel.

Example 2.2. Consider the following graphs:



Then notice that $S_2 = \widehat{R_2}(v_1 \hookrightarrow v_2)$ and $S_2 = R_2^2(v_2 \hookrightarrow v_1)$.

Recall that a graph E satisfies Condition (L) in case every cycle in E has an exit. It is clear that if E satisfies Condition (L), then so also does $E(w \hookrightarrow v)$ for any shift graph constructed from E.

We are now in position to prove the first of two "Change the Graph" Theorems.

Theorem 2.3. *Let E be a row-finite graph, and let $v \neq w \in E^0$ be vertices which are not sinks. If there exists an injection*

$$\theta : s^{-1}(w) \to s^{-1}(v)$$

such that $r(e) = r(\theta(e))$ for every $e \in s^{-1}(w)$, then $L(E(w \hookrightarrow v))$ is a homomorphic image of $L(E)$. Moreover, if either:

(1) *E satisfies Condition* (L)*, or*
(2) *the field K is infinite,*

then there exists a K-algebra isomorphism $\varphi : L(E) \to L(E(w \hookrightarrow v))$. (*The isomorphism φ is not an isomorphism of $\mathbb{Z}$-graded K-algebras.*)

Proof. Let $F = E(w \hookrightarrow v)$, and let $s_E^{-1}(w) = \{e_1, \ldots, e_n\}$. Given any $e_i \in s_E^{-1}(w)$, we define in $L(F)$ the element

$$T_{e_i} = f_{v,w} e_i.$$

Notice that $T_{e_i} \neq 0$ for every $e_i \in s_E^{-1}(w)$, and that $T_{e_i} \neq T_{e_j}$ whenever $i \neq j$. Now consider the subalgebra A of $L(F)$ generated by

$$\left\{v, e, e^*, T_{e_i}, T_{e_i}^* \;\middle|\; v \in E^0,\ e \in E^1 \setminus \theta\big(s_E^{-1}(w)\big),\ e_i \in s_E^{-1}(w)\right\}.$$

Then, if $i \neq j$, we have

$$T_{e_i}^* T_{e_j} = (f_{v,w} e_i)^* (f_{v,w} e_j) = e_i^* f_{v,w}^* f_{v,w} e_j = e_i^* e_j = 0 = \theta(e_i)^* \theta(e_j),$$

while $T_{e_i}^* T_{e_i} = e_i^* e_i = r(e_i) = r(\theta(e_i)) = \theta(e_i)^* \theta(e_i)$. Also, $s(T_{e_i}) = s(f_{v,w}) = v = s(\theta(e_i))$ and $r(T_{e_i}) = r(e_i) = r(\theta(e_i))$. Moreover, the only generators in A starting in v which do not belong to $s^{-1}(v) \setminus \theta(s^{-1}(w))$ are of the form T_{e_i} with $e_i \in s_E^{-1}(w)$. Thus,

$$\sum_{\{e\in s^{-1}(v)\setminus\theta(s^{-1}(w))\}} ee^* + \sum_{i=1}^{n} T_{e_i}T_{e_i}^* = \sum_{\{e\in s^{-1}(v)\setminus\theta(s^{-1}(w))\}} ee^* + \sum_{i=1}^{n} f_{v,w}e_i e_i^* f_{v,w}^*$$
$$= \sum_{\{e\in s^{-1}(v)\setminus\theta(s^{-1}(w))\}} ee^* + f_{v,w}\left(\sum_{i=1}^{n} e_i e_i^*\right) f_{v,w}^*$$
$$= \sum_{\{e\in s^{-1}(v)\setminus\theta(s^{-1}(w))\}} ee^* + f_{v,w}f_{v,w}^* = v.$$

Hence the generators of A satisfy the same relations as do the elements of the set $\{v, e, e^* \mid v \in E^0,\ e \in E^1\}$ in $L(E)$. Thus by the Universal Homomorphism Property of $L(E)$ there exists a unique algebra morphism extending the natural bijection

$$\begin{array}{rcl} \varphi\colon L(E) & \to & L(F), \\ e & \mapsto & e, \\ g & \mapsto & T_{\theta^{-1}(g)}, \\ e^* & \mapsto & e^*, \\ g^* & \mapsto & T^*_{\theta^{-1}(g)}, \\ v & \mapsto & v \end{array}$$

for every $e \in E^1 \setminus \theta(s_E^{-1}(w))$, every $g \in \theta(s_E^{-1}(w))$, and every $v \in E^0$.

Since $e_i \in E^1 \setminus \theta(s_E^{-1}(w))$ for every $e_i \in s_E^{-1}(w)$, we have $e_i \in A$, whence

$$f_{v,w} = f_{v,w}w = f_{v,w}\sum_{i=1}^{n} e_i e_i^* = \sum_{i=1}^{n} T_{e_i}e_i^*.$$

But $e_i \in s_E^{-1}(w)$ implies $T_{e_i} = T_{\theta^{-1}(\theta(e_i))} = \varphi(\theta(e_i))$, so that

$$f_{v,w} = \sum_{i=1}^{n} T_{e_i}e_i^* = \varphi\left(\sum_{i=1}^{n}\theta(e_i)e_i^*\right),$$

and hence φ is onto.

We note here that φ is not a graded homomorphism, since $\deg(g) = 1$, while $\deg(\varphi(g)) = \deg(T_{\theta^{-1}(g)}) = 2$. Thus Lemma 1.1 does not apply in this situation.

In the first case, if E satisfies Condition (L), then the injectivity of φ may be established by Lemma 1.2.

For the second case, if K is infinite, then for every $t \in K^*$ we can define the automorphism α_t of $L_K(F)$ by the extension of

$$\begin{array}{rcl} \alpha_t\colon L(F) & \to & L(F), \\ e & \mapsto & te, \\ f_{v,w} & \mapsto & f_{v,w}, \\ e^* & \mapsto & t^{-1}e^*, \\ f_{v,w}^* & \mapsto & f_{v,w}^*, \\ v & \mapsto & v \end{array}$$

for every $e \in E^1 \setminus \theta(s_E^{-1}(w))$ and every $v \in E^0$. In this way we get an action $\alpha : K^* \to \mathrm{Aut}_K(L_K(F))$. It is straightforward to check that, for every $t \in K^*$, $\varphi \circ \tau_t^F = \alpha_t \circ \varphi$, where τ_t^F is the gauge action of K^* on $L_K(F)$. Thus, the injectivity of φ derives from the AGIUT (Theorem 1.8). □

Example 2.4. Recall the graphs in Example 2.2. On the one side, $S_2 = \widehat{R_2}(v_1 \hookrightarrow v_2)$, whence $L(\widehat{R_2}) \cong L(S_2)$ by Theorem 2.3. On the other side, $S_2 = R_2^2(v_2 \hookrightarrow v_1)$, so that $L(S_2) \cong L(R_2^2)$ again by Theorem 2.3.

Theorem 2.3 admits a corresponding statement in the context of Cuntz–Krieger graph C^*-algebras. As far as we know, no such analogous result has been established elsewhere in the C^*-algebra literature. We do so here.

Corollary 2.5. *Let E be a row-finite graph, and let $v \neq w \in E^0$ be vertices which are not sinks. If there exists an injection*

$$\theta : s^{-1}(w) \to s^{-1}(v)$$

such that $r(e) = r(\theta(e))$ for every $e \in s^{-1}(w)$, then $C^*(E) \cong C^*(E(w \hookrightarrow v))$.

Proof. We will follow the C^*-algebra notation (see e.g. [8]). Let $F = E(w \hookrightarrow v)$. Given any $e \in s_E^{-1}(w)$, we define in $C^*(F)$ the element

$$T_e = s_{f_{v,w}} s_e e.$$

Notice that $T_e \neq 0$ for every $e \in s_E^{-1}(w)$, and that $T_e \neq T_f$ whenever $e \neq f \in s_E^{-1}(w)$. Now consider the C^*-subalgebra A of $C^*(F)$ generated by

$$\mathcal{S} = \bigl\{p_v, s_e, T_g \bigm| v \in E^0,\ e \in E^1 \setminus \theta\bigl(s_E^{-1}(w)\bigr),\ g \in s_E^{-1}(w)\bigr\}.$$

To simplify notation, let $s_E^{-1}(w) = \{e_1, \ldots, e_n\}$. Then, the same argument as in the proof of Theorem 2.3 shows that $\mathcal{S}$ is a Cuntz–Krieger E-family, whence there exists a unique C^*-algebra morphism extending the natural bijection

$$\begin{array}{rccc} \varphi : & C^*(E) & \to & C^*(F), \\ & s_e & \mapsto & s_e, \\ & s_g & \mapsto & T_{\theta^{-1}(g)}, \\ & p_v & \mapsto & p_v \end{array}$$

for every $e \in E^1 \setminus \theta(s_E^{-1}(w))$, every $g \in \theta(s_E^{-1}(w))$, and every $v \in E^0$. The same argument as above shows that φ is an onto map, while injectivity is a consequence of Gauge-Invariant Uniqueness Theorem for graph C^*-algebras (see e.g. [13, Theorem 2.2]), applied to the $\mathbb{T}$-action on $C^*(F)$ defined by

$$\begin{array}{rcl} \alpha_z: C^*(F) & \to & C^*(F), \\ s_e & \mapsto & z s_e, \\ s_{f_{v,w}} & \mapsto & s_{f_{v,w}}, \\ p_v & \mapsto & p_v \end{array}$$

for every $z \in \mathbb{T}$. □

In our second main result of this section, we show how to "unbundle" specific sets of edges, and subsequently replace these unbundled sets by a collection of new edges and new vertices. The following definition is borrowed from [8, Section 3].

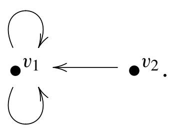
Definition 2.6. Let $E = (E^0, E^1, r, s)$ be a row-finite graph. For each $v \in E^0$ which is not a sink, partition $s^{-1}(v)$ into disjoint nonempty subsets $\mathcal{E}_v^1, \dots, \mathcal{E}_v^{m(v)}$, where $m(v) \geqslant 1$. (If v is a sink, then we put $m(v) = 0$.) Let $\mathcal{P}$ denote the resulting partition of E^1. We form the *out-split graph* $E_s(\mathcal{P})$ *from* E *using* $\mathcal{P}$ as follows: Let

$$E_s(\mathcal{P})^0 = \{v^i \mid v \in E^0,\ 1 \leqslant i \leqslant m(v)\} \cup \{v\colon m(v) = 0\},$$
$$E_s(\mathcal{P})^1 = \{e^j \mid e \in E^1,\ 1 \leqslant j \leqslant m(r(e))\} \cup \{e\colon m(r(e)) = 0\},$$

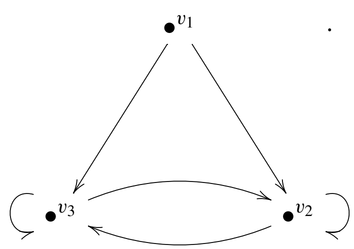
and define $r_{E_s(\mathcal{P})}, s_{E_s(\mathcal{P})} : E_s(\mathcal{P})^1 \to E_s(\mathcal{P})^0$ for $e \in \mathcal{E}_{s(e)}^i$ by

$$s_{E_s(\mathcal{P})}(e^j) = s(e)^i \quad \text{and} \quad s_{E_s(\mathcal{P})}(e) = s(e)^i,$$
$$r_{E_s(\mathcal{P})}(e^j) = r(e)^j \quad \text{and} \quad r_{E_s(\mathcal{P})}(e) = r(e).$$

Example 2.7. Consider the graph

R_2^2: 

Let $\mathcal{P}$ be the partition of the edges of R_2^2 containing only one edge per subset. Then the out-split graph of R_2^2 using $\mathcal{P}$ is

E: 

Similar to the graph C^*-algebra case, we get an isomorphism result for the Leavitt path algebras of out-split graphs. This result is the second of our two "Change the Graph" Theorems.

Theorem 2.8. *(See [8, Theorem 3.2].) Let E be a row-finite graph, $\mathcal{P}$ a partition of E^1 and $E_s(\mathcal{P})$ the out-split graph formed from E using $\mathcal{P}$. Then there is an isomorphism of $\mathbb{Z}$-graded K-algebras $\pi : L(E) \to L(E_s(\mathcal{P}))$.*

Proof. The proof is essentially the same as that given in [8, Theorem 3.2], except when showing the injectivity of the homomorphism. We include the argument here for the sake of completeness.

Given $v \in E^0$ and $e \in E^1$, set $Q_v = v$ if $m(v) = 0$, $T_e = e$ if $m(r(e)) = 0$,

$$Q_v = \sum_{1 \leqslant i \leqslant m(v)} v^i \quad \text{if } m(v) \neq 0 \quad \text{and} \quad T_e = \sum_{1 \leqslant j \leqslant m(r(e))} e^j \quad \text{if } m\bigl(r(e)\bigr) \neq 0.$$

Since E is row-finite, all of these sums are finite. We claim that $\{T_e, Q_v \mid e \in E^1,\ v \in E^0\}$ is a family in $L(E_s(\mathcal{P}))$ satisfying the same relations as $\{v, e \mid v \in E^0,\ e \in E^1\}$.

The collection $\{Q_v \mid v \in E^0\}$ is a set of nonzero mutually orthogonal idempotents (since the Q_v are sums of idempotents satisfying the same properties). The elements T_e for $e \in E^1$ clearly satisfy $T_e^* T_f = 0$ whenever $e \neq f$, because they consist of sums of elements with the same property. For $e \in E^1$ it is easy to see that $T_e^* T_e = Q_{r(e)}$.

For $e \in E^1$ with $m(r(e)) \neq 0$, since $r_{E_s}(e^j) \neq r_{E_s}(e^k)$, for $j \neq k$, we have

$$T_e T_e^* = \Bigl(\sum_{1 \leqslant j \leqslant m(r(e))} e^j\Bigr)\Bigl(\sum_{1 \leqslant k \leqslant m(r(e))} e^k\Bigr)^* = \sum_{1 \leqslant j \leqslant m(r(e))} e^j e^{j*}. \tag{1}$$

If $m(r(e)) = 0$ then $T_e T_e^* = ee^*$. For $v \in E^0$ and $1 \leqslant i \leqslant m(v)$ put

$$\mathcal{E}^i_{1,v} = \bigl\{e \in \mathcal{E}^i_v \bigm| m\bigl(r(e)\bigr) \geqslant 1\bigr\} \quad \text{and} \quad \mathcal{E}^i_{0,v} = \bigl\{e \in \mathcal{E}^i_v \bigm| m\bigl(r(e)\bigr) = 0\bigr\}.$$

If $v \in E^0$ is not a sink then $s^{-1}(v) = \bigcup_{i=1}^{m(v)} \mathcal{E}^i_v$ and for $1 \leqslant i \leqslant m(v)$ we have

$$s^{-1}_{E_s(\mathcal{P})}\bigl(v^i\bigr) = \bigl\{e^j \bigm| e \in \mathcal{E}^i_{1,v},\ 1 \leqslant j \leqslant m\bigl(r(e)\bigr)\bigr\} \cup \mathcal{E}^i_{0,v}.$$

Hence using (1) we may compute

$$\begin{aligned} Q_v = \sum_{1 \leqslant i \leqslant m(v)} v^i &= \sum_{1 \leqslant i \leqslant m(v)} \sum_{e \in \mathcal{E}^i_{1,v}} \sum_{1 \leqslant j \leqslant m(r(e))} e^j e^{j*} + \sum_{1 \leqslant i \leqslant m(v)} \sum_{e \in \mathcal{E}^i_{0,v}} ee^* \\ &= \sum_{1 \leqslant i \leqslant m(v)} \sum_{e \in \mathcal{E}^i_v} T_e T_e^* = \sum_{\{e \mid s(e) = v\}} T_e T_e^*, \end{aligned}$$

completing the proof of our claim, since vertices $v \in E^0$ with $m(v) = 0$ are sinks.

Then, by the Universal Homomorphism Property of $L(E)$ there is a homomorphism $\pi : L(E) \to L(E_s(\mathcal{P}))$ taking e to T_e, e^* to T_e^* and v to Q_v. To prove that π is onto we show that the generators of $L(E_s(\mathcal{P}))$ lie in $L(T_e, Q_v)$, the subalgebra of $L(E_s(P))$ generated by $\{T_e, T_e^*, Q_v\}$. Suppose that $w = v^j \in E_s(\mathcal{P})^0$ is not a sink, set $e \in \mathcal{E}^j_v$, and pick $1 \leqslant k \leqslant m(r(e))$. Then $\{f \in E_s(\mathcal{P})^1 \mid s_{E_s(\mathcal{P})}(f) = v^j\} = \bigcup_{e \in \mathcal{E}^j_v} \{e^k \mid 1 \leqslant k \leqslant m(r(e))\}$, and we have

$$v^j = \left(\sum_{\{f \in E_s(\mathcal{P})^1 | s_{E_s(\mathcal{P})}(f) = v^j\}} ff^* \right) = \left(\sum_{\{e \in \mathcal{E}_v^j\}} \sum_{\{1 \leqslant k \leqslant m(r(e))\}} e^k e^{k*} \right) = \left(\sum_{\{e \in \mathcal{E}_v^j\}} T_e T_e^* \right).$$

If w is a sink, then $w = Q_w$. Thus, $w \in L(T_e, Q_v)$.

If $e^j \in E_s(\mathcal{P})^1$ then $m(r(e)) \neq 0$. Since $r(e)^j \in L(T_e, Q_v)$ we have $e^j = T_e r(e)^j \in L(T_e, Q_v)$.

If $e \in E_s(\mathcal{P})^1$ then $m(r(e)) = 0$ and so $e = T_e \in L(T_e, Q_v)$.

Since Q_v is a sum of vertices and T_e is a sum of edges, we get that π is a $\mathbb{Z}$-graded map, whence the injectivity of π is guaranteed by Lemma 1.1, and the result follows. □

There is one specific partition which will play an important role throughout the sequel.

Definition 2.9. For any row-finite graph E, the *maximal out-splitting* $\tilde{E}$ of E is formed by using the partition having $m(v) = |s^{-1}(v)|$ for every $v \in E^0$ which is not a sink. In other words, $\tilde{E}$ is the graph formed from E by using the partition $\mathcal{P}$ of E^1 which admits no refinements.

Corollary 2.10. *Let E be a row-finite graph, and let $\tilde{E}$ denote the maximal out-splitting of E. Then $L_K(E) \cong L_K(\tilde{E})$ as $\mathbb{Z}$-graded K-algebras.*

As it turns out, the maximal out-splitting $\tilde{E}$ for a graph $E = (E^0, E^1, r, s)$ without sinks is isomorphic to a graph which is well known among graph theorists. Recall that the *dual graph* of a graph E is the graph $\widehat{E} = (E^1, E^2, r', s')$, where $r'(ef) = f$ and $s'(ef) = e$.

Proposition 2.11. *For any row-finite graph E without sinks, the maximal out-splitting graph $\tilde{E}$ is isomorphic to the dual graph $\widehat{E}$.*

Proof. Since the out-splitting is maximal, and E is assumed to contain no sinks, we have

$$\tilde{E}^0 = \{v^e : s(e) = v\} \quad \text{and} \quad \tilde{E}^1 = \{e^f : s(f) = r(e)\}.$$

The maps $v^e \mapsto e$ and $e^f \mapsto ef$ are easily shown to induce an isomorphism from $\tilde{E}$ to $\widehat{E}$. □

As a consequence of this proposition, it is reasonable to define the dual graph $\widehat{E}$ of *any* row-finite graph E to be its maximal out-splitting graph $\tilde{E}$. Thus, by Corollary 2.10, we get the following algebraic analog to a well-known result for graph C^*-algebras.

Corollary 2.12. *If E is any row-finite graph, then $L(E) \cong L(\widehat{E})$ as $\mathbb{Z}$-graded K-algebras.*

3. The purely infinite simple unital case

In this section we apply results from Section 2 to obtain information about the collection of purely infinite simple unital Leavitt path algebras. Our first goal is to establish Corollary 3.5, which shows that, up to isomorphism, all purely infinite simple unital Leavitt path algebras arise from a well-behaved subset of finite graphs. We start by reminding the reader of the germane ring- and graph-theoretic ideas.

Definitions 3.1. Let R be a ring. A nonzero idempotent $e \in R$ is *infinite* if the right ideal eR contains a proper direct summand isomorphic to eR. A ring R is *purely infinite simple* if

(1) R is simple (i.e., R contains no proper two-sided ideals), and
(2) every right ideal of R contains an infinite idempotent.

Definitions 3.2. Let E be a row-finite graph. If $v, w \in E^0$, we say w *connects to* v if there is a path μ in E for which $s(\mu) = w$ and $r(\mu) = v$. If c is a cycle in E, we say w *connects to* c if w connects to some vertex v in c. A subset $H \subseteq E^0$ is *hereditary* if whenever $w \in H$ and $v \in E^0$ and w connects to v, then $v \in H$. The set H is *saturated* if whenever $s^{-1}(v) \neq \emptyset$ and $\{r(e)\colon s(e) = v\} \subseteq H$, then $v \in H$. The graph E is called *cofinal* if the only hereditary saturated subsets of E^0 are $\emptyset$ and E^0.

The purely infinite simple Leavitt path algebras have been described in [2, Theorem 11]. Specifically, $L(E)$ is purely infinite simple if and only if (i) E is cofinal, (ii) E satisfies Condition (L), and (iii) every vertex in E^0 connects to a cycle.

Definition 3.3. Given a row-finite graph E, we say that E satisfies *Condition* (Sing) if E contains no parallel edges. Rephrased, E satisfies Condition (Sing) if for every pair of vertices $v, w \in E^0$, $\mathrm{card}(\{e \in E^1 \mid s(e) = v \text{ and } r(e) = w\}) \leqslant 1$.

Proposition 3.4. *Let E be a finite graph such that $L(E)$ is a purely infinite simple ring. If $k = \mathrm{card}(E^1)$, then for every $n \geqslant k$ there exists a graph E_n such that*:

(1) $\mathrm{card}(E_n^0) = n$,
(2) *E_n satisfies Condition* (Sing), *and*
(3) $L(E_n) \cong L(E)$.

Proof. By induction on n. We start by establishing the result in the case $n = k$. By Corollary 2.10 we obtain $E_n = \tilde{E}$ having $\mathrm{card}(E_n^0) = \mathrm{card}(E^1) = k$ and $L(E_n) \cong L(E)$. Since E_n is the maximal out-splitting of E, it clearly satisfies Condition (Sing).

Suppose that the result holds for some $n \geqslant k$; we will prove that it holds for $n + 1$. So we pick E_n satisfying Condition (Sing), $\mathrm{card}(E_n^0) = n$ and $L(E_n) \cong L(E)$. Since $L(E_n)$ is purely infinite simple, E_n contains at least one cycle having an exit by [2, Theorem 11]. Thus, there exists a vertex $v \in E_n^0$ such that $\mathrm{card}(s^{-1}(v)) \geqslant 2$. Consider any nontrivial partition $\mathcal{P}$ with $s^{-1}(v) = \mathcal{E}_v^1 \cup \mathcal{E}_v^2$ arbitrary, and for any $w \in E_n^0 \setminus \{v\}$ which is not a sink, let $s^{-1}(w) = \mathcal{E}_w^1$ be the trivial partition. Then the out-split graph of E_n by the partition $\mathcal{P}$ satisfies $E_s(\mathcal{P})^0 = (E_n^0 \setminus \{v\}) \cup \{v^1, v^2\}$, so that $\mathrm{card}(E_s(\mathcal{P})^0) = n + 1$. Moreover, by Theorem 2.8, $L(E_s(\mathcal{P})) \cong L(E_n)$. Also, as $E_s(\mathcal{P})$ is obtained by a partition of E_n, it necessarily satisfies Condition (Sing). Hence by defining $E_{n+1} = E_s(\mathcal{P})$, the induction step is established. □

Thus, in order to decide whether two purely infinite simple unital Leavitt path algebras $L(E)$ and $L(F)$ are isomorphic, it is enough to consider the problem for isomorphic algebras $L(E_n)$ and $L(F_n)$ where $|E_n^0| = |F_n^0|$ and each of E_n, F_n satisfy Condition (Sing). More formally,

Corollary 3.5. *An invariant $\mathcal{K}$ classifies purely infinite simple unital Leavitt path algebras up to isomorphism if and only if, for each $n \in \mathbb{N}$, $\mathcal{K}$ classifies up to isomorphism purely infinite simple unital Leavitt path algebras of graphs having n vertices which satisfy Condition* (*Sing*)*.*

In fact, we can extend Proposition 3.4 to stipulate that the new graphs have no sources.

Proposition 3.6. *Let E be a finite graph such that $L(E)$ is a purely infinite simple ring. Then there exists $n_0 \geqslant |E^1|$ such that, for any $n \geqslant n_0$ there exists a finite graph F such that*:

(1) $|F^0| = n$,
(2) *F satisfies Condition* (*Sing*),
(3) *F has no sources, and*
(4) $L(E) \cong L(F)$.

(*We note that the isomorphism in* (4) *is not necessarily $\mathbb{Z}$-graded.*)

Proof. By Proposition 3.4, for any $n \geqslant |E^1|$, there exists a graph E_n with n vertices satisfying Condition (Sing) such that $L(E) \cong L(E_n)$. We show now that we can modify E_n if necessary to produce a graph F for which F has the desired properties. Let C denote the set of vertices of E_n^0 which lie in a closed simple path, and set $T = E_n^0 \setminus C$. Notice that if $T \neq \emptyset$, $v \in T$ and v is not a source, then $s(r^{-1}(v)) \subset T$. For, suppose $w \in s(r^{-1}(v)) \cap C$. Then, there exists a cycle μ such that $w \in \mu^0$. By cofinality of E_n, there exists $\alpha \in E_n^*$ with $s(\alpha) = v$ and $r(\alpha) \in \mu^0$. But then, $v \in C$, contradicting the assumption. Hence, if S denotes the sources of E_n^0, it is clear that every vertex in T lies in the tree of some $x \in S$, and that the tree of S feeds into C. Since E_n is finite, we can partition T in layers as follows: $T_0 = C$, and for any $k \geqslant 1$, $T_k = \{v \in E^0 \setminus \bigcup_{i=0}^{k-1} T_i \mid r(s^{-1}(v)) \subseteq \bigcup_{i=0}^{k-1} T_i\}$.

Now, we will prove the result by induction on k (the number of layers of T). For $k = 1$, fix $v \in T_1$, so that $r(s^{-1}(v)) \subseteq C$. Set $s^{-1}(v) = \{e_1, \dots, e_l\}$, and split this set in singletons $\mathcal{E}_1, \dots, \mathcal{E}_l$, where $\mathcal{E}_i = \{e_i\}$. Consider the out-split graph induced by this partition, say $E_s(\mathcal{P})$, and notice that in this graph the set C coincides with that of E_n, the new vertices $v^1, \dots, v^l$ lie in T_1, and the remaining vertices lie in the same layers as it did in E_n; in particular, both graphs have the same number of layers. Moreover, by Theorem 2.8, $L(E_n) \cong L(E_s(\mathcal{P}))$. Also, for each $1 \leqslant i \leqslant l$, $s^{-1}(v_i) = \{e_i\}$ and $r(e_i) \in C$, being $r(e_i), \dots, r(e_l)$ different vertices (as both E_n and $E_s(\mathcal{P})$ satisfy (Sing)).

Now, for each e_i ($1 \leqslant i \leqslant l$), let μ_i be a cycle such that $r(e_i) \in \mu_i^0$, and let $f_i \in \mu_i^1$ such that $r(f_i) = r(e_i)$. Since $s^{-1}(v_i) = \{e_i\}$, the map $\Theta_i : s^{-1}(v_i) \to s^{-1}(s(f_i))$ sending e_i to f_i is a well-defined injective map such that $r(e_i) = r(f_i)$. Now, l applications of the shift construction give us a graph

$$E_n^v = \big[\big[\cdots\big[E_s(\mathcal{P})\big(v^1 \hookrightarrow r(f_1)\big)\big]\big(v^2 \hookrightarrow r(f_2)\big)\big]\cdots\big(v^{l-1} \hookrightarrow r(f_{l-1})\big)\big]\big(v^l \hookrightarrow r(f_l)\big)$$

satisfying (Sing), where $v^1, \dots, v^l \in C$, $|T_1(E_n^v)| = (|T_1(E_n)| - 1)$, $T_s(E_n^v) = T_s(E_n)$ for every $s > 1$. As $E_s(\mathcal{P})$ satisfies Condition (L), $L(E_s(\mathcal{P})) \cong L(E_n^v)$ by Theorem 2.3.

Applying this argument recurrently on the elements of the (finite) set T_1, we construct $E_n^{(1)}$ satisfying Condition (Sing), with $L(E_n) = L(E_n^{(1)})$, and with the property that the set T in $E_n^{(1)}$ has one less layer than the corresponding in E_n. Hence, the result holds by induction.

The parenthetical remark follows from the fact that Theorem 2.3 has been used in the proof, and the isomorphism between Leavitt path algebras ensured by that result is not in general $\mathbb{Z}$-graded. □

Recall that for a ring R, we denote by $K_0(R)$ the Grothendieck group of R. This is the group F/S, where F is the free group generated by isomorphism classes of finitely generated projective left R-modules, and S is the subgroup of F generated by symbols of the form $[P \oplus Q] - [P] - [Q]$. As is standard, we denote the isomorphism class of R in $K_0(R)$ by $[1_R]$. The group $K_0(R)$ is the universal group of the monoid $V(R)$ of isomorphism classes of finitely generated projective left R-modules (with binary operation in $V(R)$ given by $[A] + [B] = [A \oplus B]$). Because the rings we consider here are purely infinite simple Leavitt path algebras, we have the following more explicit relationship between K_0 and V in this setting:

$$V\big(L(E)\big) \cong \{0\} \sqcup K_0\big(L(E)\big)$$

(see for instance [5, Corollary 2.2]).

For a row-finite graph E, the *monoid of* E, denoted M_E, is the monoid generated by the set E^0 of vertices of E modulo appropriate relations, specifically,

$$M_E = \left\langle a_v, v \in E^0 \;\middle|\; a_v = \sum_{\{e \in s^{-1}(v)\}} a_{r(e)} \right\rangle.$$

It is shown in [6, Theorem 2.5] that $V(L(E)) \cong M_E$ for any row-finite graph E. This yields $K_0(L(E)) \cong \mathrm{Grot}(M_E) := G$, where $\mathrm{Grot}(M_E)$ denotes the universal group of the monoid M_E. Since M_E is finitely generated, so is its universal group G. Thus G admits a presentation $\pi : \mathbb{Z}^n \to G$ (an epimorphism). Here $\ker(\pi)$ is the subgroup of relations, which in this setting corresponds to the image of the group homomorphism $A_E^t - I : \mathbb{Z}^n \to \mathbb{Z}^n$, where A_E^t is the transpose of the incidence matrix A_E of E. Hence we get

$$K_0\big(L(E)\big) \cong G \cong \mathbb{Z}^n / \ker(\pi) = \mathbb{Z}^n / \mathrm{im}\big(A_E^t - I\big) = \mathrm{coker}\big(A_E^t - I\big).$$

Moreover, under this isomorphism the element $[1_{L(E)}]$ is represented by $(1, 1, \ldots, 1)^t + \mathrm{im}(A_E^t - I)$ in $\mathrm{coker}(A_E^t - I)$.

Throughout the remainder of this article we seek to describe properties of the Grothendieck groups $K_0(L(E))$ for various graphs E. To do so we will use the displayed isomorphism $K_0(L(E)) \cong \mathrm{coker}(A_E^t - I)$ often, and without explicit mention. (We present some examples below which indicate how one may directly compute $\mathrm{coker}(A_E^t - I)$.)

In the study of C^*-algebras, an important role is played by the Classification Theorem of purely infinite simple unital nuclear C^*-algebras (see e.g. [9,11]). Specifically, Kirchberg and Phillips (independently) showed that if X and Y are purely infinite simple unital C^*-algebras (satisfying certain additional conditions), then $X \cong Y$ as C^*-algebras if and only if (i) $K_0(X) \cong K_0(Y)$ via an isomorphism ϕ for which $\phi([1_X]) = [1_Y]$, and (ii) $K_1(X) \cong K_1(Y)$.

As it turns out, in the more specific case of purely infinite simple unital Cuntz–Krieger graph C^*-algebras, K-theoretic information is in fact encoded in the transpose A_E^t of the incidence matrix A_E of the graph E. Specifically, when E has no sinks, then by [14, Theorem 3.9]

$$K_0\big(C^*(E)\big) \cong \operatorname{coker}\big(A_E^t - I\big) \quad \text{and} \quad K_1\big(C^*(E)\big) \cong \ker\big(A_E^t - I\big),$$

where I is the identity matrix of size $n = |E^0|$.

We seek a similar result in the setting of purely infinite simple unital Leavitt path algebras. So suppose E and F are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple unital. By [2, Theorem 11] these graphs contain no sinks. By Proposition 3.6 we can assume without loss of generality that E and F have the same number n of vertices and that they have no sources. Thus if $K_0(L(E)) \cong K_0(L(F))$, then using the previously established isomorphism we get $\operatorname{coker}(A_E^t - I) \cong \operatorname{coker}(A_F^t - I)$. This in turn implies (by the Fundamental Theorem of Finitely Generated Abelian Groups) the existence of invertible matrices $P, Q \in \mathrm{M}_n(\mathbb{Z})$ such that $A_F^t - I = P(A_E^t - I)Q$. Thus $\ker(A_F^t - I) \cong \ker(A_E^t - I)$ (as these are subgroups of $\mathbb{Z}^n$ having equal rank); notice that in particular, since $K_1(C^*(E)) \cong \ker(A_E^t - I)$, we have recovered the result of [14, Theorem 3.9] for graph C^*-algebras. Moreover, by using the unique unital ring map $\psi : \mathbb{Z} \to K$, we get that the PAQ-equivalence of $A_E^t - I$ and $A_F^t - I$ also holds on K. If $K^\times$ denotes the multiplicative group on nonzero elements in K, then the previous remark implies that $\operatorname{coker}(A_E^t - I : (K^\times)^n \to (K^\times)^n)$ and $\operatorname{coker}(A_F^t - I : (K^\times)^n \to (K^\times)^n)$ (where $A_E^t - I$ and $A_E^t - I$ are seen as multiplicative maps on $(K^\times)^n$) are also isomorphic. Since by [7, Theorem 3.19], for any finite graph G with n vertices with no sinks or sources we have

$$K_1\big(L(G)\big) \cong \operatorname{coker}\big(A_G^t - I : \big(K^\times\big)^n \to \big(K^\times\big)^n\big) \oplus \ker\big(A_G^t - I : \mathbb{Z}^n \to \mathbb{Z}^n\big),$$

we conclude that the hypothesis $K_0(L(E)) \cong K_0(L(F))$ in fact yields $K_1(L(E)) \cong K_1(L(F))$ as a consequence. With this observation and the aforementioned Kirchberg–Phillips result in mind, it is then natural to ask the following.

The Classification Question for purely infinite simple unital Leavitt path algebras. Suppose E and F are graphs for which $L(E)$ and $L(F)$ are purely infinite simple unital. If $K_0(L(E)) \cong K_0(L(F))$ via an isomorphism ϕ having $\phi([1_{L(E)}]) = [1_{L(F)}]$, must $L(E)$ and $L(F)$ be isomorphic?

Much of the remainder of this article is taken up in addressing The Classification Question. We notationally abbreviate the statement

$$K_0\big(L(E)\big) \cong K_0\big(L(F)\big) \quad \text{via an isomorphism } \phi \text{ having } \phi\big([1_{L(E)}]\big) = [1_{L(F)}]$$

by writing

$$\big(K_0\big(L(E)\big), [1_{L(E)}]\big) \cong \big(K_0\big(L(F)\big), [1_{L(F)}]\big).$$

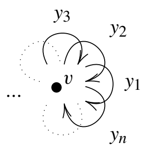
Definition 3.7. We recall that for each integer $n \geqslant 2$, the *Leavitt algebra* L_n is the free associative K-algebra with generators $\{x_i, y_i \colon 1 \leqslant i \leqslant n\}$ and relations

$$(1) \quad x_i y_j = \delta_{ij} \quad \text{for all } 1 \leqslant i, j \leqslant n, \quad \text{and} \quad (2) \quad \sum_{i=1}^{n} y_i x_i = 1.$$

See [1] or [10] for additional information about L_n. In particular, the isomorphism

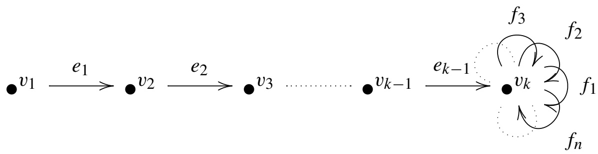
$$L_n \cong L(R_n)$$

follows immediately, where R_n is the "rose with n petals" graph



.

For $n \geqslant 2$ and $k \geqslant 1$ we define the graph B_n^k to be



.

Then by [2, Proposition 13] we have

$$L\big(B_n^k\big) \cong \mathrm{M}_k(L_n).$$

We will use this isomorphism throughout the sequel, often without explicit mention.

By [5, Theorem 4.2] we have that $K_0(L_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$. In fact, it is clear from this isomorphism that $(K_0(L_n), [1_{L_n}]) \cong (\mathbb{Z}/(n-1)\mathbb{Z}, \bar{1})$. Because K_0 is a Morita invariant, we also necessarily have $K_0(\mathrm{M}_k(L_n)) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ for any $k \in \mathbb{N}$. It is straightforward to show that this isomorphism gives

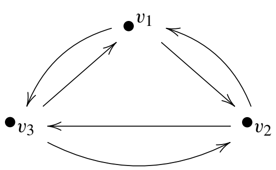
$$\big(K_0\big(\mathrm{M}_k(L_n)\big), [1_{\mathrm{M}_k(L_n)}]\big) \cong \big(\mathbb{Z}/(n-1)\mathbb{Z}, \bar{k}\big).$$

We will revisit this isomorphism later, in two regards. First, we will show in Example 3.9 that it can be re-established using tools from Leavitt path algebras. Second, we will establish in Section 4 an affirmative answer to The Classification Question among a specific class of Leavitt path algebras $L(E)$, to wit, if $(K_0(L(E)), [1_{L(E)}]) \cong (\mathbb{Z}/(n-1)\mathbb{Z}, \bar{k})$, then $L(E) \cong \mathrm{M}_k(L_n)$.

We now present some examples in which we explicitly compute $\operatorname{coker}(A_E^t - I)$ for various graphs E. Additional examples and computations of this type can be found in [14, p. 32 and Example 3.31].

Example 3.8. Consider the graph

E_1^6:



.

We compute $\operatorname{coker}(A^t_{E_1^6} - I)$. First,

$$A^t_{E_1^6} - I = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Then, applying the classical PAQ-reduction, we get that $A^t_{E_1^6} - I$ is equivalent to the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

while the invertible matrix P, which fixes the basis change in the arrival free group, is

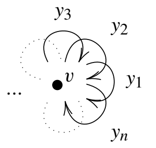
$$P = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then,

$$K_0\big(L\big(E_1^6\big)\big) \cong \operatorname{coker}(D) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

On the other side, as in $\operatorname{coker}(A^t_{E_1^6} - I)$ the element $[1_{L(E_1^6)}]$ is represented by $(1, 1, 1)^t$, applying the change of basis we get that the image of the element $[1_{L(E_1^6)}]$ in $\operatorname{coker}(D)$ is $P \cdot (1, 1, 1)^t = (-1, 2, 2)^t$, modulo the relation defined by $\operatorname{im}(D)$, so that we conclude that $[1_{L(E_1^6)}]$ corresponds to $(\bar{0}, \bar{0})$.

Example 3.9. Consider the graph R_n

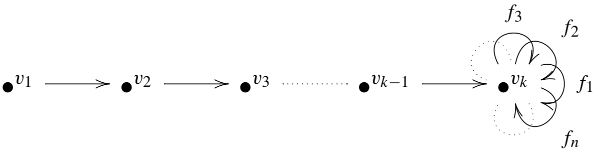


and recall that $L(R_n) \cong L_n$. We will use the K_0-picture described above to compute $K_0(L_n)$. We first compute $\operatorname{coker}(A^t_{R_n} - I)$. This is obvious, as

$$A^t_{R_n} - I = (n - 1)$$

whence $K_0(L_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$. Since this matrix is in reduced form, $[1_{L_n}]$ corresponds to $\bar{1} \in \mathbb{Z}/(n-1)\mathbb{Z}$.

Now consider the graph B_n^k



and recall that $L(B_n^k) \cong \mathrm{M}_k(L_n)$. First,

$$A^t_{B_n^k} - I = \begin{pmatrix} n-1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Then, applying the classical PAQ-reduction, we get that $A^t_{B_n^k} - I$ is equivalent to the diagonal matrix

$$D = \begin{pmatrix} n-1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

while the invertible matrix P, which fixes the basis change in the arrival free group, is

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Then $K_0(L(B_n^k)) \cong \operatorname{coker}(D) = \mathbb{Z}/(n-1)\mathbb{Z}$. (We have thereby re-established a previously observed isomorphism between $K_0(L(B_n^k))$ and $\mathbb{Z}/(n-1)\mathbb{Z}$.) On the other side, as in $\operatorname{coker}(A^t_{B_n^k} - I)$ the element $[1_{B_n^k}]$ is represented by $(1, 1, \ldots, 1)^t$, applying the change of basis we get that the image of the element $[1_{L(B_n^k)}]$ in $\operatorname{coker}(D)$ is $P \cdot (1, 1, \ldots, 1)^t = (k, -(k-1), -(k-2), \ldots, -2, -1)^t$, modulo the relation defined by $\operatorname{im}(D)$, so that we conclude that $[1_{L(B_n^k)}]$ corresponds to $\bar{k}$.

4. Graphs

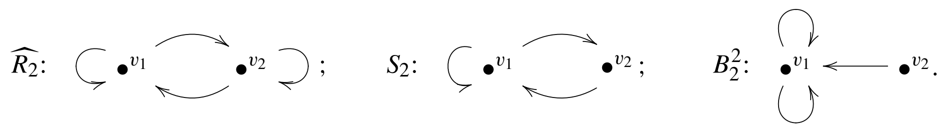
In this section we will show how to use Theorems 2.3 and 2.8 in order to classify purely infinite simple unital Leavitt path algebras according to their K_0-data. Specifically, we give an affirmative answer to The Classification Question for Leavitt path algebras coming from various collections of graphs.

Among these collections will be graphs whose K_0-data matches the K_0-data for Leavitt path algebras of the form $L(E) \cong \mathrm{M}_k(L_n)$. (For instance, as noted previously, by [2, Proposition 13] the graph B_n^k has $L(B_n^k) \cong \mathrm{M}_k(L_n)$.) That is, we will have Leavitt path algebras $L(E)$ for which $(K_0(L(E)), [1_{L(E)}]) \cong (\mathbb{Z}/(n-1)\mathbb{Z}, \bar{k})$. For such collections we provide additional evidence that The Classification Question has an affirmative answer, by showing that the relevant Leavitt path algebras are indeed isomorphic to $\mathrm{M}_k(L_n)$ for appropriate n, k. This is shown at the end of each of the germane subsections into which the section is divided.

We note that all of the graphs we consider throughout this section satisfy Condition (L) (since the graphs arise in the context of purely infinite simple Leavitt path algebras). Thus Theorem 2.3 applies to all of the shift graph constructions produced here, regardless of the size of the field K.

4.1. Graphs with two vertices

We start by analyzing graphs having two vertices, which satisfy Condition (Sing), and for which the associated Leavitt path algebra is purely infinite simple. Concretely, they are the following:



Using the description of $(K_0(L(E)), [1_{L(E)}])$ given at the end of Section 3, it is straightforward to show that each of these three graphs has $(K_0(L(E)), [1_{L(E)}]) \cong (\{\bar{0}\}, \bar{0})$. But then the isomorphisms between the respective algebras can be found in Examples 2.4 and 2.7.

Thus we have answered in the affirmative a specific case of The Classification Question for purely infinite simple unital Leavitt path algebras.

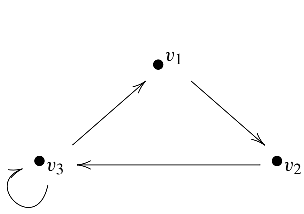
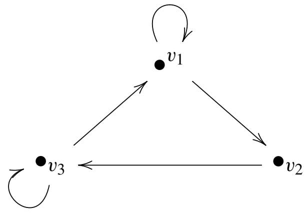
Proposition 4.1. *Suppose E and F are graphs having Condition (Sing), for which $L(E)$ and $L(F)$ are purely infinite simple unital, and $|E^0| = |F^0| = 2$. If $K_0(L(E)) \cong K_0(L(F))$ via an isomorphism ϕ for which $\phi([1_{L(E)}]) = [1_{L(F)}]$, then $L(E) \cong L(F)$.*

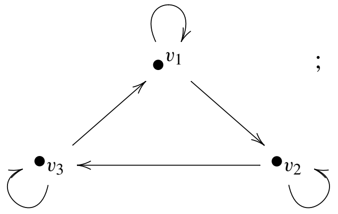
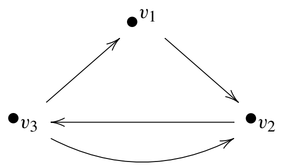
The three graphs of Proposition 4.1 each have K_0-data $(\{\bar{0}\}, \bar{0})$, which matches the K_0-data of $\mathrm{M}_1(L_2) \cong L_2$. We show that in fact $L(E) \cong L_2$ for each of these three graphs. This will follow directly from the isomorphism $L_2 \cong L(B_2^2)$ ensured by Propositions 5.2 and 5.3 below.

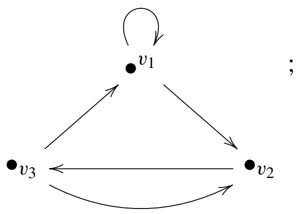
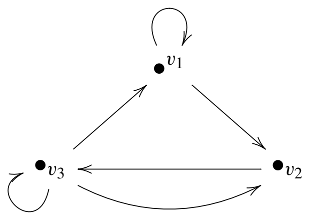
4.2. Graphs with three vertices

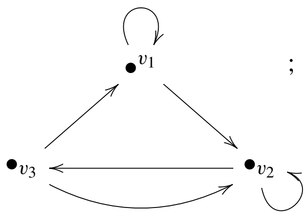
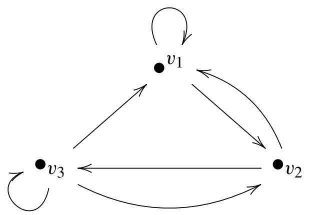
We continue by analyzing graphs having three vertices, which satisfy Condition (Sing), and for which the associated Leavitt path algebra is purely infinite simple. It turns out there exist 34 such graphs. Unlike the previously analyzed situation for graphs with two vertices, there will be more than one pair of the form $(K_0(L(E)), [1_{L(E)}])$ arising from this collection. (There are seven such pairs, to be exact.) We partition all 34 of these graphs along the seven K_0-data pairs, and then use the tools of Section 2 to show that the Leavitt path algebras within each equivalence class are indeed pairwise isomorphic. Throughout we use without mention the description of $(K_0(L(E)), [1_{L(E)}])$ presented at the end of Section 3.

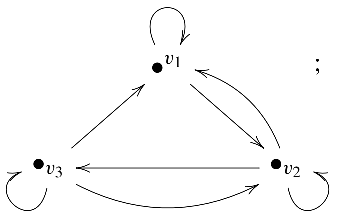
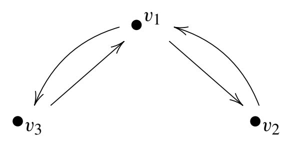
1. $(K_0(L(E)), [1_{L(E)}]) \cong (\{\bar{0}\}, \bar{0})$: In this situation we have 18 graphs, listed as follows:

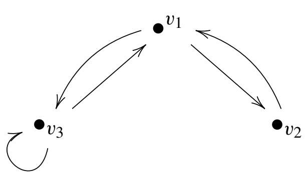
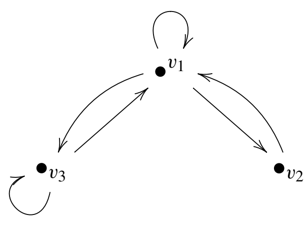
E_1^1:  ; E_2^1:  ;

E_3^1:  ; E_4^1:  ;

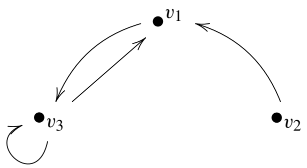
E_5^1:  ; E_6^1:  ;

E_7^1:  ; E_8^1:  ;

E_9^1:  ; E_{10}^1:  ;

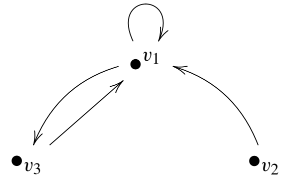
E_{11}^1:  ; E_{12}^1:  ;

E^1_{13}:



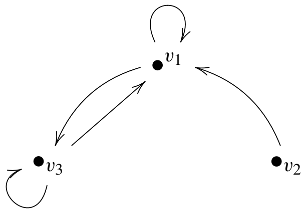
;

E^1_{14}:



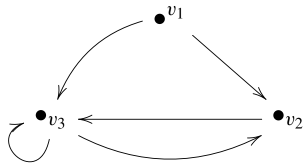
;

E^1_{15}:



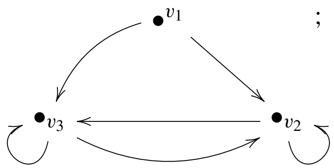
;

E^1_{16}:



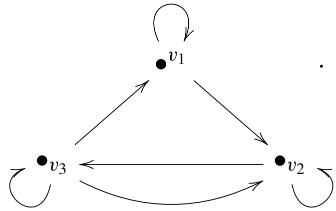
;

E^1_{17}:



;

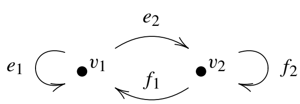
E^1_{18}:



.

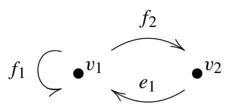
Now, we prove the isomorphisms as follows: First consider the out-splitting of

$\widehat{R_2}$:



partitioning the edges in $\mathcal{P} = \{e_1\} \cup \{e_2\} \cup \{f_1, f_2\}$, and notice that $(\widehat{R_2})_s(\mathcal{P}) = E^1_6$. Thus, $L(\widehat{R_2}) \cong L(E^1_6)$ by Theorem 2.8. Now, consider the out-splitting of

S_2:



partitioning the edges in $\mathcal{P} = \{e_1\} \cup \{f_1\} \cup \{f_2\}$, and notice that $(S_2)_s(\mathcal{P}) = E^1_5$. Thus, $L(S_2) \cong L(E^1_5)$ by Theorem 2.8. Finally, the maximal out-splitting of B^2_2 equals E^1_{17}, whence $L(B^2_2) \cong L(\widehat{B^2_2}) = L(E^1_{17})$ by Corollary 2.12. Since $L(S_2) \cong L(\widehat{R_2}) \cong L(B^2_2)$ by Proposition 4.1, we have shown that $L(E^1_5) \cong L(E^1_6) \cong L(E^1_{17})$, which in turn can be used to verify the isomorphisms with all the remaining indicated Leavitt path algebras by noticing that

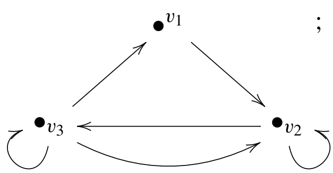
(1) $E^1_{17}(v_3 \hookrightarrow v_2) = E^1_{16}$,
(2) $E^1_{16}(v_3 \hookrightarrow v_1) = E^1_4$,
(3) $E^1_{16}(v_1 \hookrightarrow v_3) = E^1_{14}$,
(4) $E^1_2(v_2 \hookrightarrow v_3) = E^1_5$,
(5) $E^1_1(v_2 \hookrightarrow v_3) = E^1_4$,

(6) $E_5^1(v_3 \hookrightarrow v_1) = E_{10}^1$,
(7) $E_7^1(v_1 \hookrightarrow v_3) = E_2^1$,
(8) $E_7^1(v_3 \hookrightarrow v_1) = E_{11}^1$,
(9) $E_9^1(v_3 \hookrightarrow v_2) = E_6^1$,
(10) $E_9^1(v_2 \hookrightarrow v_3) = E_{12}^1$,
(11) $E_8^1(v_1 \hookrightarrow v_3) = E_7^1$,
(12) $E_{14}^1(v_3 \hookrightarrow v_2) = E_{13}^1$,
(13) $E_{15}^1(v_1 \hookrightarrow v_3) = E_{14}^1$,
(14) $E_{18}^1(v_2 \hookrightarrow v_3) = E_7^1$,
(15) $E_{18}^1(v_1 \hookrightarrow v_3) = E_3^1$.

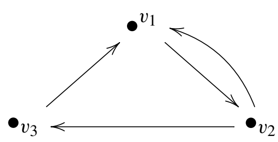
Then, all those Leavitt path algebras are pairwise isomorphic by Theorem 2.3, so we are done.

The eighteen graphs of this subsection have K_0-data $(\{\bar{0}\}, \bar{0})$. But the purely infinite simple Leavitt path algebra $L(B_2^2) \cong L_2$ has this same K_0-data as well. As further evidence of an affirmative answer to The Classification Question, we note that indeed we have shown, for all eighteen graphs E in this subsection, that $L(E) \cong L_2$. (We established the isomorphism $L(B_2^2) \cong L(E_{17}^1)$ in the course of the proof.)

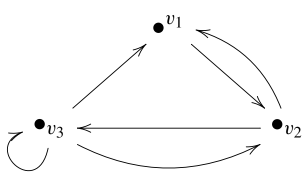
2. $(K_0(L(E)), [1_{L(E)}]) \cong (\mathbb{Z}/2\mathbb{Z}, \bar{0})$: In this situation we have 6 graphs, listed as follows:

E_1^2: 

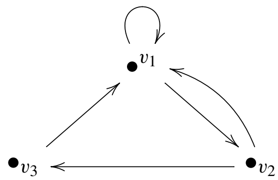
;

E_2^2: 

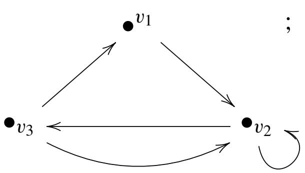
;

E_3^2: 

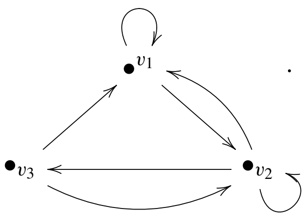
;

E_4^2: 

;

E_5^2: 

;

E_6^2: 

.

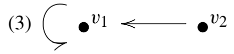
Now, we prove the isomorphisms as follows: First notice that

(1) $E_6^2(v_3 \hookrightarrow v_1) = E_3^2$,

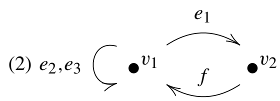
(2) $E_6^2(v_1 \hookrightarrow v_3) = E_1^2$,
(3) $E_1^2(v_2 \hookrightarrow v_3) = E_5^2$,
(4) $E_4^2(v_1 \hookrightarrow v_3) = E_5^2$,
(5) $E_5^2(v_1 \hookrightarrow v_2) = E_2^2$.

Then, all those Leavitt path algebras are pairwise isomorphic by Theorem 2.3, so we are done.

The six graphs of this subsection have K_0-data $(\mathbb{Z}/2\mathbb{Z}, \bar{0}) = (\mathbb{Z}/2\mathbb{Z}, \bar{2})$. But the purely infinite simple Leavitt path algebra $\mathrm{M}_2(L_3)$ has this same K_0-data as well. As further evidence of an affirmative answer to The Classification Question, we now show, for all six graphs E in this subsection, that $L(E) \cong \mathrm{M}_2(L_3)$. To see this, by [2, Proposition 13] we have $\mathrm{M}_2(L_3) \cong L(B_3^2)$, where

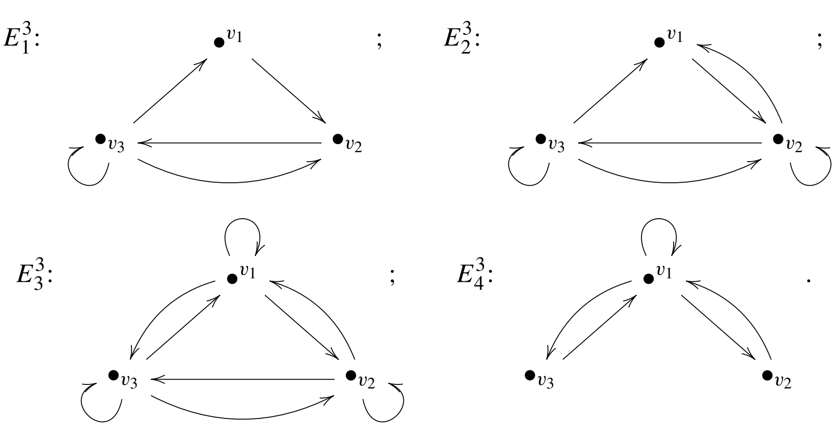
B_3^2: 

(here the notation (n) indicates that there are n parallel edges), and a single application of Theorem 2.3 gives us $E_1 = B_3^2(v_1 \hookrightarrow v_2)$, where

E_1: 

Partitioning the edges in $\mathcal{P} = \{f\} \cup \{e_1, e_2\} \cup \{e_3\}$, we get $(E_1)_s(\mathcal{P}) = E_6^2$, so that the result holds by Theorem 2.8, as desired.

3. $(K_0(L(E)), [1_{L(E)}]) \cong (\mathbb{Z}/2\mathbb{Z}, \bar{1})$: In this situation we have 4 graphs, listed as follows:

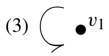


Now, we prove the isomorphisms as follows: First notice that

(1) $E_3^3(v_2 \hookrightarrow v_1) = E_2^3$,
(2) $E_2^3(v_2 \hookrightarrow v_3) = E_4^3$,
(3) $E_4^3(v_3 \hookrightarrow v_2) = E_1^3$.

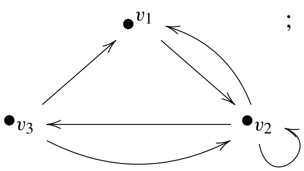
Then, all those Leavitt path algebras are pairwise isomorphic by Theorem 2.3, so we are done.

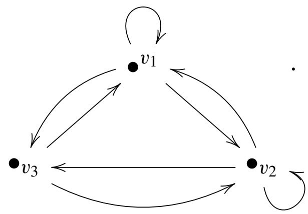
The four graphs of this subsection have K_0-data $(\mathbb{Z}/2\mathbb{Z}, \bar{1})$. But the purely infinite simple Leavitt path algebra $\mathrm{M}_1(L_3) \cong L_3$ has this same K_0-data as well. As further evidence of an affirmative answer to The Classification Question, we now show, for all four graphs E in this subsection, that $L(E) \cong L_3$. To see this, recall that $L_3 \cong L(R_3)$, where

R_3: 

Notice that the maximal out-splitting of R_3 equals E_3^3, whence by Theorem 2.8 we get the desired result.

4. $(K_0(L(E)), [1_{L(E)}]) \cong (\mathbb{Z}/3\mathbb{Z}, \bar{1})$: In this situation we have 2 graphs, listed as follows:

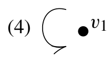
E_1^4: 

; E_2^4: 

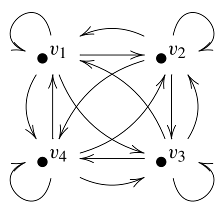
.

By noticing that $E_2^4(v_2 \hookrightarrow v_1) = E_1^4$, these Leavitt path algebras are isomorphic by Theorem 2.3, so we are done.

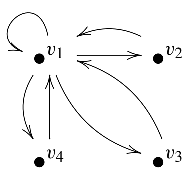
The two graphs of this subsection have K_0-data $(\mathbb{Z}/3\mathbb{Z}, \bar{1})$. But the purely infinite simple Leavitt path algebra $\mathrm{M}_1(L_4) \cong L_4$ has this same K_0-data as well. As further evidence of an affirmative answer to The Classification Question, we now show, for both graphs E in this subsection, that $L(E) \cong L_4$. For, recall that $L_4 \cong L(R_4)$, where

R_4: 

Consider the maximal out-splitting $\widehat{R_4}$ of R_4

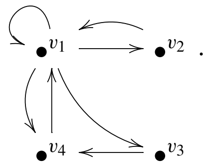
$\widehat{R_4}$: 

Then, $L(R_4) \cong L(\widehat{R_4})$ by Corollary 2.12. Consider the graph $\widehat{E} = [[\widehat{R_4}(v_1 \hookrightarrow v_2)](v_1 \hookrightarrow v_3)](v_1 \hookrightarrow v_4)$, that is,

$\widehat{E}$: 

Hence, $L(\widehat{R_4}) \cong L(\widehat{E})$ by several applications of Theorem 2.3. If we separate the edges in E_1^4 in such way that the edges emitted by v_3 are divided in two singletons, then by defining $F = (E_1^4)_s(\mathcal{P})$, we get

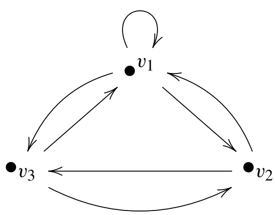
F:



Clearly, $F = \widehat{E}(v_3 \hookrightarrow v_4)$. Thus, the result holds by Theorem 2.8.

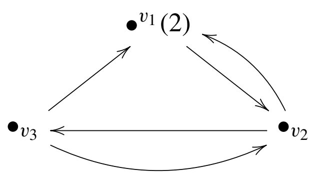
5. $(K_0(L(E)), [1_{L(E)}]) \cong (\mathbb{Z}/4\mathbb{Z}, \bar{2})$: In this situation we have one graph, listed as follows:

E_1^5:



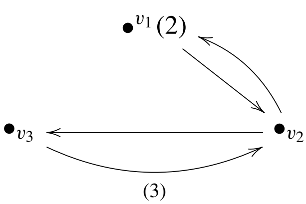
The one graph of this subsection has K_0-data $(\mathbb{Z}/4\mathbb{Z}, \bar{2})$. But the purely infinite simple Leavitt path algebra $\mathrm{M}_2(L_5)$ has this same K_0-data as well. As further evidence of an affirmative answer to The Classification Question, we now show, for the graph E in this subsection, that $L(E) \cong \mathrm{M}_2(L_5)$. For, let $E_1 = E_1^5(v_2 \hookrightarrow v_1)$,

E_1:



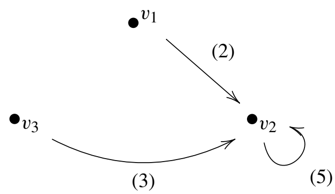
(here (2) means that there are two edges from v_1 to v_2). Then, $E_1 = E_2(v_1 \hookrightarrow v_3)$, where

E_2:



Also, $E_2 = [E_3(v_3 \hookrightarrow v_2)](v_1 \hookrightarrow v_2)$, where

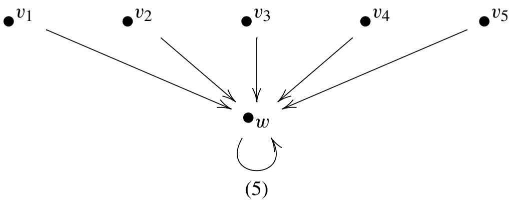
E_3:



.

Partitioning the set of edges emitted by v_1 and v_2 in singletons, we get

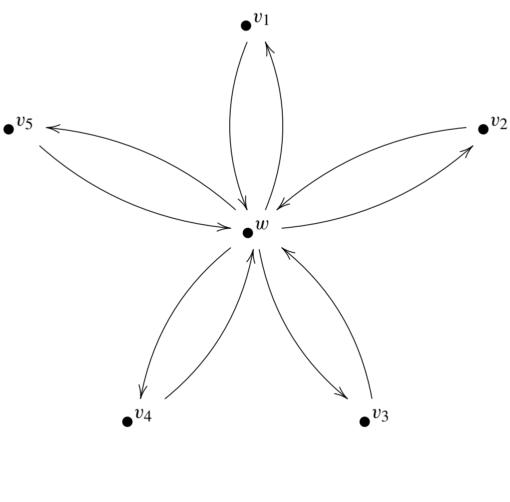
E_4:



.

Thus, $E_5 = [[[[E_4(v_1 \hookrightarrow w)](v_2 \hookrightarrow w)](v_3 \hookrightarrow w)](v_4 \hookrightarrow w)](v_5 \hookrightarrow w)$ is the graph

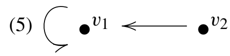
E_5:



.

Recall that $\mathrm{M}_2(L_5) \cong L(B_5^2)$, where

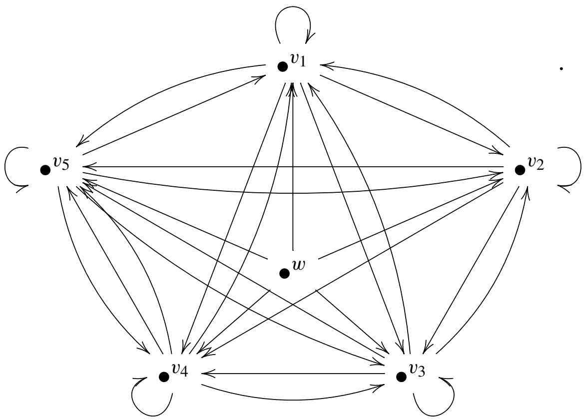
B_5^2:



.

Consider $\widehat{B_5^2}$

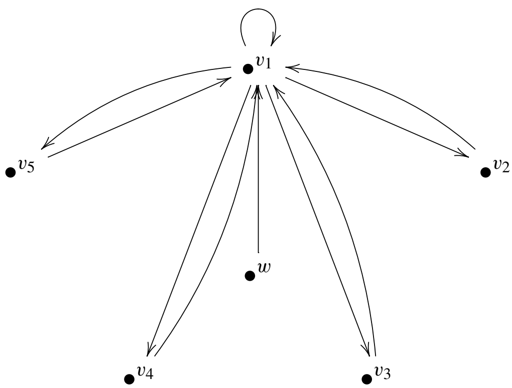
$\widehat{B_5^2}$:



.

We have $L(B_5^2) \cong L(\widehat{B_5^2})$ by Corollary 2.12. Then, we get $F_1 = [[[[[\widehat{B_5^2}](v_1 \hookrightarrow v_2)](v_1 \hookrightarrow v_3)](v_1 \hookrightarrow v_4)](v_1 \hookrightarrow v_5)](v_1 \hookrightarrow w)$

F_1:

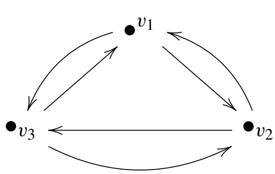


and hence $L(\widehat{B_5^2}) \cong L(F_1)$ by several applications of Theorem 2.3. Finally, $E_5 = F_1(w \hookrightarrow v_1)$. Thus, $L(E_5) \cong L(F_1)$ by Theorem 2.3, as desired.

In the final two subsections we analyze the remaining three graphs. Since the K_0-data of these graphs is not of the form $(\mathbb{Z}/(n-1)\mathbb{Z}, \bar{k})$, connections between the Leavitt path algebras of these three graphs and algebras of the form $\mathrm{M}_k(L_n)$ are not of issue.

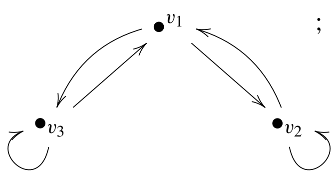
6. $(K_0(L(E)), [1_{L(E)}]) \cong (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, (\bar{0}, \bar{0}))$: In this situation we have 1 graph, listed as follows:

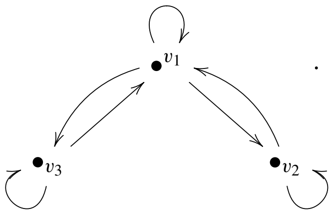
E_1^6:



.

7. $(K_0(L(E)), [1_{L(E)}]) \cong (\mathbb{Z}, 0)$: In this situation we have 2 graphs, listed as follows:

E_1^7: 

; E_2^7: 

.

By noticing that $E_2^7(v_3 \hookrightarrow v_1) = E_1^7$, these Leavitt path algebras are isomorphic by Theorem 2.3, so we are done.

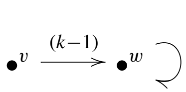
Thus we have answered in the affirmative another specific case of The Classification Question for purely infinite simple unital Leavitt path algebras.

Proposition 4.2. *Suppose E and F are graphs satisfying Condition (Sing) for which $L(E)$ and $L(F)$ are purely infinite simple unital, and $|E^0| = |F^0| = 3$. If $K_0(L(E)) \cong K_0(L(F))$ via an isomorphism ϕ for which $\phi([1_{L(E)}]) = [1_{L(F)}]$, then $L(E) \cong L(F)$. Moreover, for any such graph E for which $(K_0(L(E)), [1_{L(E)}]) \cong (\mathbb{Z}/(n-1)\mathbb{Z}, \bar{k})$, then in fact we have $L(E) \cong \mathrm{M}_k(L_n)$.*

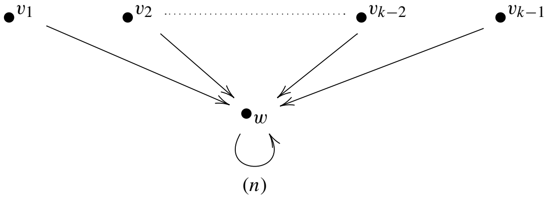
5. Isomorphisms between Leavitt algebras and their matrices

In this final section we deal with the problem of determining values of k and n for which $L_n \cong \mathrm{M}_k(L_n)$. This problem was completely solved by three of the authors in [4] using combinatorial arguments, where we show that the necessary and sufficient condition for such an isomorphism is that g.c.d.$(k, n-1) = 1$. As it turns out, one direction of this implication was already established in [10], where Leavitt shows that L_n and $\mathrm{M}_k(L_n)$ are not isomorphic whenever g.c.d.$(k, n-1) > 1$. Thus throughout this section we consider only situations in which g.c.d.$(k, n-1) = 1$. Our goal here is to prove the isomorphism $L_n \cong \mathrm{M}_k(L_n)$, using arguments afforded by Theorems 2.3 and 2.8. The graph and combinatorial approaches to this isomorphism question are essentially independent.

First, fix graphs

R_n^k: 

;

A_n^k: 

;

B_n^k: 

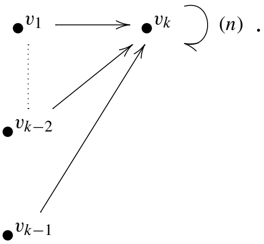
.

Then, we have the following result.

Lemma 5.1. *For each $n \geqslant 1$, the algebras $L(A_n^k)$, $L(B_n^k)$ and $L(R_n^k)$ are isomorphic. Moreover, each of these algebras is isomorphic to the matrix algebra $\mathrm{M}_k(L_n)$.*

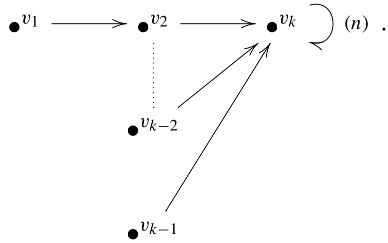
Proof. It is clear that A_n^k is obtained from R_n^k by splitting $s^{-1}(v)$ in $k-1$ singletons, whence $L(R_n^k) \cong L(A_n^k)$ by Theorem 2.8. On the other side, rewrite A_n^k as

B_{k-1}^1:



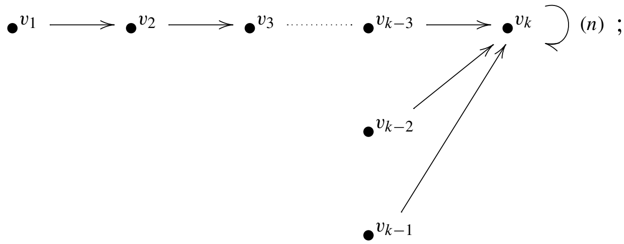
Then, $B_{k-2}^2 = B_{k-1}^1(v_2 \hookrightarrow v_1)$, where

B_{k-2}^2:

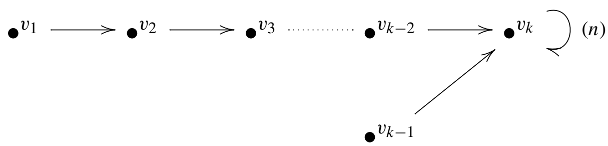


Hence, $L(B_{k-1}^1) \cong L(B_{k-2}^2)$ by Theorem 2.3. (Note that Condition (L) holds trivially.) Recurrence on this argument produces a chain of graphs whose Leavitt path algebras are isomorphic. This chain ends in

B_2^{k-2}:

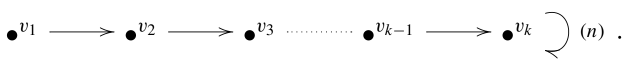


B_1^{k-1}:



with the final step being

B_n^k:



This yields the asserted isomorphisms between the three indicated Leavitt path algebras. The final statement follows immediately from [2, Proposition 13], in which the isomorphism $L(B_n^k) \cong \mathrm{M}_k(L_n)$ is established. □

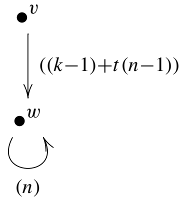
Since any graph having Leavitt path algebra isomorphic to $\mathrm{M}_k(L_n)$ must satisfy Condition (L) (as any such algebra is purely infinite simple), Theorem 2.3 may be invoked in all situations throughout the sequel. Specifically, all of the isomorphism results of this section hold regardless of the cardinality of the field K.

Here is our first result about isomorphisms between matrix rings over Leavitt algebras.

Proposition 5.2. *For every $t \geqslant 0$, for every $k \geqslant 1$ and for every $n \geqslant 2$,*

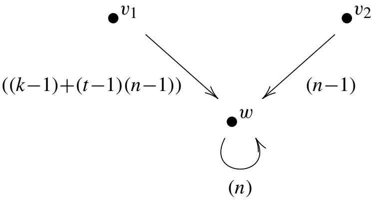
$$\mathrm{M}_k(L_n) \cong \mathrm{M}_{k+t(n-1)}(L_n).$$

Proof. We will prove the result by induction on t. The case $t = 0$ being clear, we suppose then the result holds for $t-1$. By [2, Proposition 13] we have $\mathrm{M}_{k+t(n-1)}(L_n) \cong L(R_n^{k+t(n-1)})$, where

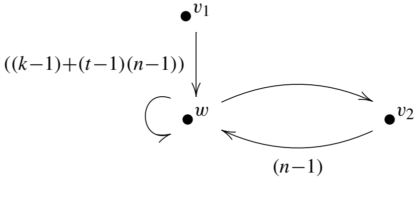
$R_n^{k+t(n-1)}$: 

.

Splitting the edges emitted by v in two sets, one with $(k-1)+t(n-1)$ edges, and the other with $(n-1)$ edges, we get $E_1 = (R_n^{k+t(n-1)})_s(\mathcal{P})$,

E_1: 

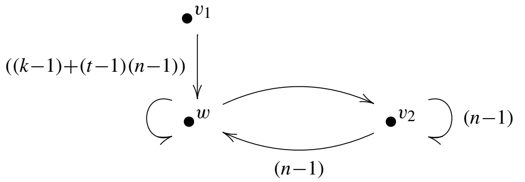
and $L(R_n^{k+t(n-1)}) \cong L(E_1)$ by Theorem 2.8. Now, let $E_2 = E_1(v_2 \hookrightarrow w)$,

E_2: 

.

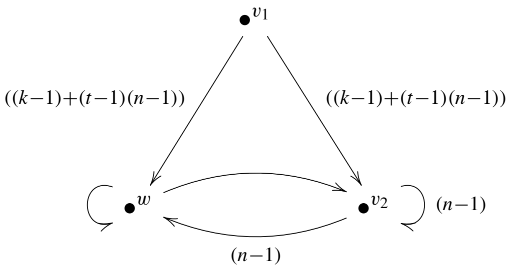
By Theorem 2.3, $L(E_1) \cong L(E_2)$. Take E_3 to be

E_3:



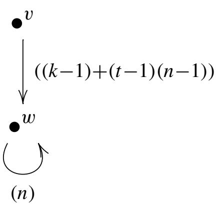
and notice that $E_2 = [\cdots[E_3(w \hookrightarrow v_2)](w \hookrightarrow v_2)]\cdots](w \hookrightarrow v_2)$, $(n-1)$ times, so that $L(E_2) \cong L(E_3)$ by Theorem 2.3. Now, take E_4

E_4:



and notice that $E_3 = [\cdots[E_4(w \hookrightarrow v_1)](w \hookrightarrow v_1)]\cdots](w \hookrightarrow v_1)$, $(k-1)+(t-1)(n-1)$ times, so that $L(E_3) \cong L(E_4)$ by Theorem 2.3. Finally, take $R_n^{k+(t-1)(n-1)}$

$R_n^{k+(t-1)(n-1)}$:



and notice that E_4 is the out-splitting of $R_n^{k+(t-1)(n-1)}$ over the edges emitted by w in two sets, one with $(n-1)$ edges, and the other a singleton. Thus, $L(E_4) \cong L(R_n^{k+(t-1)(n-1)})$; but the latter is isomorphic to $\mathrm{M}_k(L_n)$ by the induction hypothesis. Thus $\mathrm{M}_{k+t(n-1)}(L_n) \cong \mathrm{M}_k(L_n)$, which completes the induction step. □

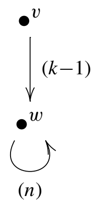
We note that the conclusion of Proposition 5.2 also follows from the fact that the free left L_n-modules of ranks k and $k+t(n-1)$ are isomorphic, so that the endomorphism rings of these modules are isomorphic, and such endomorphism rings are in turn isomorphic to the indicated matrix rings.

Here is our second result about isomorphisms between matrix rings over Leavitt algebras.

Proposition 5.3. *Let $n \geqslant 2$, $k \geqslant 2$ be such that k divides n. Then $L_n \cong \mathrm{M}_k(L_n)$.*

Proof. We have that $n = kl$ for some l. We recall again that $\mathrm{M}_k(L_n) \cong L(R_n^k)$,

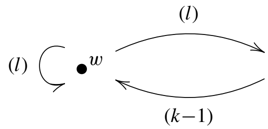
R_n^k:



.

Of course $n = (k-1)l + l$. Consider $E_1 = [\cdots[R_n^k(v \hookrightarrow w)](v \hookrightarrow w)]\cdots](v \hookrightarrow w)$, l times

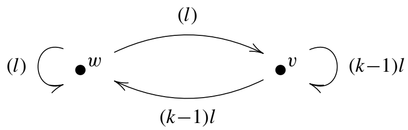
E_1:



.

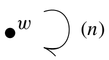
By Theorem 2.3, $L(R_n^k) \cong L(E_1)$. Now, take E_2

E_2:



and notice that $E_1 = [\cdots[E_2(w \hookrightarrow v)](w \hookrightarrow v)]\cdots](w \hookrightarrow v)$, $k-1$ times, so that $L(E_1) \cong L(E_2)$ by Theorem 2.3. Finally, consider R_n

R_n:



.

If we consider a partition of the edges emitted by w in two sets with l and $(k-1)l$ edges respectively, then $(R_n)_s(\mathcal{P}) = E_2$, so that $L(R_n) \cong L(E_2)$ by Theorem 2.8. As $L_n \cong L(R_n)$, the desired result holds. □

The final goal of this article is to use our two "Change the Graph" isomorphisms to establish the isomorphism $L_n \cong \mathrm{M}_k(L_n)$ whenever g.c.d.$(k, n-1) = 1$. As mentioned previously, this isomorphism was established in [4], using completely different techniques. As a consequence of the current discussion, we obtain yet more evidence suggesting an affirmative answer to The Classification Question.

As we shall see, establishing the isomorphism $L_n \cong \mathrm{M}_k(L_n)$ utilizes a ten step process, where each step requires the use of one or the other of the two Change the Graph Theorems. Steps 1 through 3, and 5 through 10, are relatively transparent; however, Step 4 requires some additional work, which we take care of in the next few results.

We begin by relating Theorem 2.3 to a matrix operator.

Definition 5.4. Let $M \in \mathrm{M}_p(\mathbb{Z})$. Define $\Phi_M : \mathrm{M}_p(\mathbb{Z}) \to \mathrm{M}_p(\mathbb{Z})$ by setting, for each $A \in \mathrm{M}_p(\mathbb{Z})$,

$$\Phi_M(A) = MA + (I_p - M)$$

where I_p is the identity matrix in $\mathrm{M}_p(\mathbb{Z}^+)$. It is easy to check that for $M_1, M_2, \dots, M_t \in \mathrm{M}_p(\mathbb{Z})$ we have $\Phi_{M_t} \circ \cdots \circ \Phi_{M_2} \circ \Phi_{M_1} = \Phi_{M_t \cdots M_2 M_1}$.

Straightforward matrix arithmetic yields

Lemma 5.5. *Let $k, p \in \mathbb{N}$. For integers $1 \leqslant s, t \leqslant p$ with $s \neq t$ let $K \in \mathrm{M}_p(\mathbb{Z}^+)$ denote the matrix*

$$K = I_p + k e_{st}$$

where e_{st} denotes the standard (s,t) matrix unit in $\mathrm{M}_p(\mathbb{Z}^+)$. Suppose $A = (a_{ij}) \in \mathrm{M}_p(\mathbb{Z}^+)$.

(1) *If $a_{tt} \geqslant 1$ and $a_{st} \geqslant 1$ then $\Phi_K(A) \in \mathrm{M}_p(\mathbb{Z}^+)$, and for each $1 \leqslant j \leqslant p$ we have $(\Phi_K(A))_{sj} \geqslant (\Phi_K(A))_{tj}$ and $(\Phi_K(A))_{tt} \geqslant 1$.*
(2) *In addition, if $a_{ts} \geqslant 1$ then $(\Phi_K(A))_{ss} \geqslant 1$ and $(\Phi_K(A))_{ts} \geqslant 1$.*

The proof of the next result follows directly from Lemma 5.5 and the construction presented in Theorem 2.3.

Corollary 5.6. *Let $k, p \in \mathbb{N}$. Let $K \in \mathrm{M}_p(\mathbb{Z}^+)$ denote the matrix $I_p + k e_{st}$ for some pair $1 \leqslant s, t \leqslant p$ with $s \neq t$. Suppose $A = (a_{ij}) \in \mathrm{M}_p(\mathbb{Z}^+)$ has $a_{tt} \geqslant 1$ and $a_{st} \geqslant 1$, and suppose that the associated graph E_A satisfies Condition (L). Then*

$$L(E_A) \cong L(E_{\Phi_K(A)}).$$

Definition 5.7. We identify some quantities which will be useful in the sequel. Let a, b be positive integers having g.c.d.$(a,b) = 1$. Assume $a > b > 1$. We apply the standard Euclidean algorithm to find sequences of positive integers $r_0, r_1, \dots, r_m$ and $k_1, k_2, \dots, k_m$ for which

$$r_0 = k_1 r_1 + r_2, \qquad r_1 = k_2 r_2 + r_3, \quad \dots, \quad r_{m-2} = k_{m-1} r_{m-1} + r_m$$

where $r_0 = a$, $r_1 = b$, and $r_m =$ g.c.d.$(a,b) = 1$. (Note that $m \geqslant 2$ since $b > 1$.) It will be notationally useful to add a nonstandard additional equation to the end of this list by defining

$$r_{m-1} = k_m \cdot 1 + r_{m+1}$$

(in other words, we set $k_m = r_{m-1} - 1$ and $r_{m+1} = 1$). We now define a collection of 3×3 matrices based on these sequences, by setting

$$S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} 1 & 0 & 0 \\ k_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and, for every integer p with $1 \leqslant p \leqslant m/2$,

$$S_{2p} = \begin{pmatrix} 1 & k_{2p} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot S_{2p-1}, \qquad S_{2p+1} = \begin{pmatrix} 1 & 0 & 0 \\ k_{2p+1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot S_{2p}.$$

We note that $S_i \in \mathrm{M}_p(\mathbb{Z}^+)$ and $\det(S_i) = 1$ for all $0 \leqslant i \leqslant m$.

Proposition 5.8. g.c.d.$(a, b) = 1$. *Suppose also* $b > 1$. *We use the notation as in Definition* 5.7. *We denote the specific matrix* S_m *by*

$$S_m = \begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular we have $y_2 \geqslant 1$. *Then*

(1) $x_1 b - y_1 a = 1$ *and* $x_2 b - y_2 a = -1$.
(2) $x_1 - y_1 \geqslant 1$ *and* $x_2 - y_2 \geqslant 0$.
(3) $x_1 + x_2 = a$ *and* $y_1 + y_2 = b$.

Proof. Using the matrix equations given in Definition 5.7, an easy induction argument shows, for each integer p having $1 \leqslant p \leqslant m/2$, that

$$S_{2p} \begin{pmatrix} r_1 \\ -r_0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_{2p+1} \\ -r_{2p} \\ 0 \end{pmatrix} \quad \text{and} \quad S_{2p+1} \begin{pmatrix} r_1 \\ -r_0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_{2p+1} \\ -r_{2p+2} \\ 0 \end{pmatrix}.$$

Thus in particular, using that $r_m = r_{m+1} = 1$, we get

$$S_m \begin{pmatrix} r_1 \\ -r_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

But recall that by definition we have $r_1 = b$ and $r_0 = a$, so the previous equation becomes

$$\begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

This matrix equation yields (1). The equation $x_1 - y_1 \geqslant 1$ of (2) follows immediately from (1) and the hypothesis that $a > b$. For the other part of (2), note that $-1 = x_2 b - y_2 a < x_2 b - y_2 b$ (since $a > b > 0$ and $y_2 > 0$), so $-1 < (x_2 - y_2)b \in \mathbb{Z}$, so necessarily $x_2 - y_2 \geqslant 0$.

Now recall that $\det(S_i) = 1$ for all $0 \leqslant i \leqslant m$, so in particular we have $\det(S_m) = 1$, so that $x_1 y_2 - y_1 x_2 = 1$. We incorporate the previous two pieces of information in the single matrix equation

$$\begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b & -y_1 & 0 \\ -a & x_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

But then by using this same information it is easy to check that

$$\begin{pmatrix} b & -y_1 & 0 \\ -a & x_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x_1 & y_1 & 0 \\ a & b & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that right multiplying gives

$$\begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & y_1 & 0 \\ a & b & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & 0 \\ a - x_1 & b - y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular $x_2 = a - x_1$ and $y_2 = b - y_1$, from which (3) follows immediately. □

Were our aim solely to achieve a number-theoretic result regarding the Euclidean algorithm, we would have constructed analogous S-matrices inside $\mathrm{M}_2(\mathbb{Z}^+)$ rather than inside $\mathrm{M}_3(\mathbb{Z}^+)$, since clearly all of the germane computations take place in the upper 2×2 blocks of the S_i. However, as we shall see below, our one application of the combinatorial facts provided in Proposition 5.8 will be in the context of 3×3 matrices, so we choose to formulate the result accordingly.

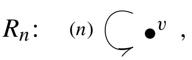
We are now in position to demonstrate our main isomorphism result, a result which will lead to verification of another piece of The Classification Question.

Theorem 5.9. *Let n, d be positive integers having g.c.d.$(d, n-1) = 1$, and let K be any field. Then $L_n \cong \mathrm{M}_d(L_n)$.*

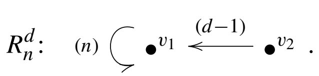
Proof. The result is trivial for $d = 1$. Now suppose $d = 2$. Then either n is even, whence d divides n and so the result holds by Proposition 5.3, or n is odd, whence g.c.d.$(d, n-1) = 2$, contradicting the hypothesis. So we may assume that $d \geqslant 3$. Also, by Proposition 5.2 we can assume that $d \leqslant n - 2$. Now write $n = dt + r$ with $0 \leqslant r \leqslant d - 1$. (In particular, $d - r + 1 \geqslant 2$.) If $r = 0$ then d divides n, so that the result holds by Proposition 5.3. If $r = 1$ then $d =$ g.c.d.$(d, n-1)$, contradicting the hypothesis. So we may also assume that $r \geqslant 2$. In particular we have $t + r - 1 \geqslant 1$ for all $t \geqslant 0$.

It will be clear that each of the graphs encountered in this proof satisfies Condition (L). Thus Theorem 2.3 may be invoked throughout, without regard to the size of the field of scalars.

Recall that $L_n \cong L(R_n)$ where R_n is the graph



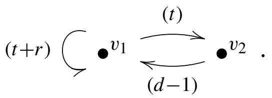
while $\mathrm{M}_d(L_n) \cong L(R_n^d)$ where R_n^d is the graph



We establish the desired result by building a ten step sequence of isomorphisms which starts with $L(R_n^d)$ and ends with $L(R_n)$.

Step 1. Consider $E_1 = R_n^d(v_2 \overset{t \text{ times}}{\hookrightarrow} v_1)$,

E_1:

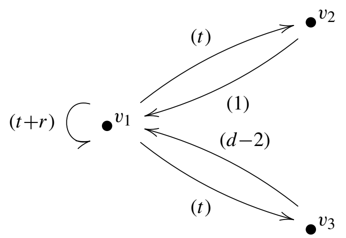


Then

$$L\big(R_n^d\big) \cong L(E_1) \quad \text{by Theorem 2.3.}$$

Step 2. Splitting the set of edges of E_1 emitted by v_2 in two sets of 1 and $d-2$ edges respectively, we get $E_2 = (E_1)_s(P)$

E_2:



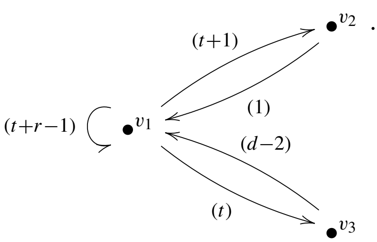
whence

$$L(E_1) \cong L(E_2) \quad \text{by Theorem 2.8.}$$

(Note that $d \geqslant 3$ guarantees that the quantity $d-2$ is nonnegative.)

Step 3. Consider $E_3 = E_2(v_2 \hookrightarrow v_1)$

E_3:



Then we get

$$L(E_2) \cong L(E_3) \quad \text{by Theorem 2.3.}$$

Step 4. We are now in position to use the number-theoretic results described above. Let A denote the matrix A_{E_3} of the graph E_3; that is,

$$A = \begin{pmatrix} t+r-1 & t+1 & t \\ 1 & 0 & 0 \\ d-2 & 0 & 0 \end{pmatrix},$$

so that $E_A = E_3$.

Since g.c.d.$(d, r-1) = 1$ we have g.c.d.$(d, d-r+1) = 1$. Let $r_0 = a = d$, $r_1 = b = d-r+1$. Note that $a > b$ (since $r \geqslant 2$) and $b > 1$ (since $r \leqslant d-1$). So we may apply the analysis given in Definition 5.7 to the pair $a = d, b = d-r+1$ to produce the indicated matrix S_m.

Now define, for $1 \leqslant i \leqslant m$,

$$K_i = I_3 + k_i e_{21} \quad \text{for } i \text{ odd}, \quad \text{and} \quad K_i = I_3 + k_i e_{12} \quad \text{for } i \text{ even}.$$

Then by construction we have $S_m = K_m \cdots K_2 K_1$.

Because $K_1 = I_3 + k_1 e_{21}$ and $a_{11} = t + r - 1 \geqslant 1$ and $a_{21} = 1$, we have by Lemma 5.5(1) that $\Phi_{K_1}(A) \in \mathrm{M}_3(\mathbb{Z}^+)$. Moreover, since $a_{12} = t+1 \geqslant 1$, Lemma 5.5(2) yields $(\Phi_{K_1}(A))_{22} \geqslant 1$ and $(\Phi_{K_1}(A))_{12} \geqslant 1$. Thus Lemma 5.5 may be applied at each step, and we thereby conclude that the matrix B defined by

$$B = \Phi_{K_m} \circ \Phi_{K_{m-1}} \circ \cdots \circ \Phi_{K_1}(A) = \Phi_S(A)$$

is in $\mathrm{M}_p(\mathbb{Z}^+)$. Now define $E_4 = E_B$ for the matrix B. So we have

$$L(E_3) \cong L(E_B) = L(E_4) \quad \text{by an application of Corollary 5.6 } m \text{ times.}$$

Prior to moving on to Step 5, we actually compute the values of the entries of the matrix $B = (b_{ij})$. By definition we have

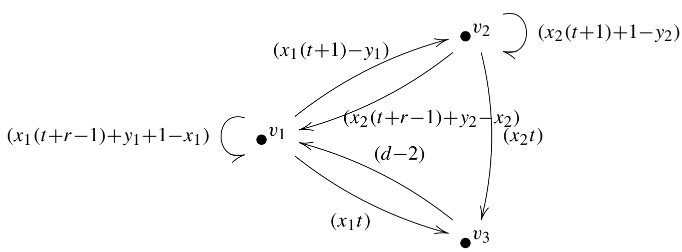
$$B = \Phi_{S_m}(A) = \begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t+r-1 & t+1 & t \\ 1 & 0 & 0 \\ d-2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1-x_1 & -y_1 & 0 \\ -x_2 & 1-y_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So upon doing the matrix arithmetic we get

$$B = \begin{pmatrix} x_1(t+r-1) + y_1 + 1 - x_1 & x_1(t+1) - y_1 & x_1 t \\ x_2(t+r-1) + y_2 - x_2 & x_2(t+1) + 1 - y_2 & x_2 t \\ d-2 & 0 & 0 \end{pmatrix}.$$

A pictorial description of $E_4 = E_B$ is then given by

E_4:



.

Step 5. By Lemma 5.8 we have $x_1 - y_1 \geqslant 1$ and $x_2 - y_2 \geqslant 0$. So we may define the graph E_5 by setting

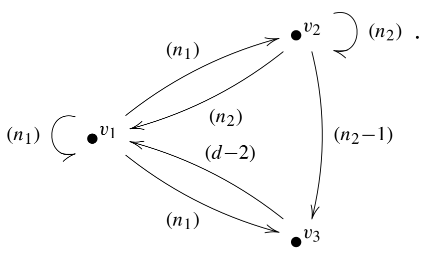
$$E_5 = \big[E_4\big(\overset{(x_1-y_1)\text{ times}}{v_3 \hookrightarrow v_1}\big)\big]\big(\overset{(x_2-y_2)\text{ times}}{v_3 \hookrightarrow v_2}\big).$$

Using the information presented in the various parts of Lemma 5.8, it is tedious but straightforward to now verify each of the following equations:

$$\begin{aligned}
b_{11} - (x_1 - y_1)b_{31} &= x_1(t+1) - y_1, \\
b_{12} - (x_1 - y_1)b_{32} &= x_1(t+1) - y_1, \\
b_{13} - (x_1 - y_1)b_{33} + (x_1 - y_1) &= x_1(t+1) - y_1, \\
b_{21} - (x_2 - y_2)b_{31} &= x_2(t+1) + 1 - y_2, \\
b_{22} - (x_2 - y_2)b_{32} &= x_2(t+1) + 1 - y_2, \\
b_{23} - (x_2 - y_2)b_{33} + (x_2 - y_2) &= x_2(t+1) - y_2.
\end{aligned}$$

For notational convenience we define $n_1 = x_1(t+1) - y_1$ and $n_2 = x_2(t+1) + 1 - y_2$. Note that $n_1 \geqslant 1$ and $n_2 \geqslant 1$ as a consequence of Lemma 5.8(2). Now using this list of equations, we have that E_5 is the graph

E_5:



In particular,

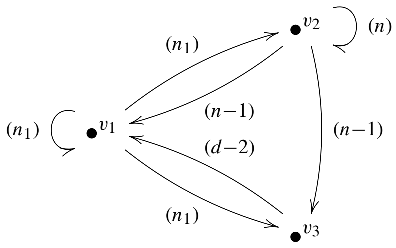
$$L(E_4) \cong L(E_5) \quad \text{by Theorem 2.3.}$$

Step 6. Since

$$\begin{aligned}
n_1 + n_2 &= (x_1 + x_2)(t+1) + 1 - (y_1 + y_2) \\
&= a(t+1) + 1 - b \\
&= dt + d + r - d \\
&= dt + r = n,
\end{aligned}$$

if we define the graph E_6 to be

E_6:

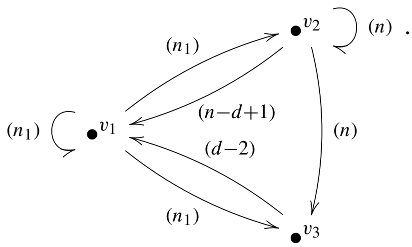


then we get $E_5 = E_6(v_1 \hookrightarrow v_2)$, so that

$$L(E_5) \cong L(E_6) \quad \text{by Theorem 2.3.}$$

Step 7. Define $E_7 = E_6(v_3 \hookrightarrow v_2)$. That is,

E_7:

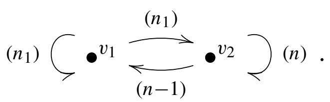


Thus

$$L(E_6) \cong L(E_7) \quad \text{by Theorem 2.3.}$$

Step 8. We let E_8 denote the graph

E_8:



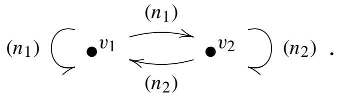
We split the set of edges emitted by v_2 in two sets of $d-2$ and $2n-d+1$ edges respectively. That is, we use the partition

Then it is not hard to see that $(E_8)_s(P) = E_7$, so that

$$L(E_7) \cong L(E_8) \quad \text{by Theorem 2.8.}$$

Step 9. Now consider $E_9 = E_8(v_1 \hookrightarrow v_2)$. Pictorially,

E_9:

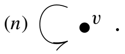


But then

$$L(E_8) \cong L(E_9) \quad \text{by Theorem 2.3.}$$

Step 10. Finally, recall that the graph R_n is given by

R_n:



If we take a partition of the set of edges emitted by v in two sets of n_1 and n_2 edges respectively, then $(R_n)_s(P) = E_9$, so that

$$L(E_9) \cong L(R_n) \quad \text{by Theorem 2.8.}$$

Thus Steps 1 through 10 yield $L(R_n^d) \cong L(R_n)$, so that we have established the isomorphism

$$\mathrm{M}_d(L_n) \cong L\big(R_n^d\big) \cong L(R_n) \cong L_n,$$

and we are done. □

We have seen above that Theorem 2.3 has a direct analog for C^*-algebras in Corollary 2.5, and that Theorem 2.8 has a direct analog for C^*-algebras in [8, Theorem 3.2]. The proof of Theorem 5.9 follows from Theorem 2.3, Theorem 2.8, and from purely combinatorial arguments (arguments which therefore hold in both the Leavitt path algebra and Cuntz–Krieger C^*-algebra settings). So the results of this section have provided a graph-theoretic approach to the following.

Theorem 5.10. *Let n, d be positive integers having g.c.d.$(d, n-1) = 1$. Then $\mathcal{O}_n \cong \mathrm{M}_d(\mathcal{O}_n)$.*

As promised, we now show how Theorem 5.9 yields the answer to another piece of The Classification Question. This same conclusion was drawn in [4]; for completeness, we present here some of the details of the proof provided there.

Theorem 5.11. *(See [4, Theorem 5.2].) Let $\mathcal{L}$ denote the set of purely infinite simple K-algebras*

$$\big\{\mathrm{M}_d(L_n) \,\big|\, d, n \in \mathbb{N}\big\}.$$

Let $B, B' \in \mathcal{L}$. Then $B \cong B'$ if and only if there is an isomorphism $\phi : K_0(B) \to K_0(B')$ for which $\phi([1_B]) = [1_{B'}]$.

Proof. It is well known that any unital isomorphism $f : B \to B'$ induces a group isomorphism $K_0(f) : K_0(B) \to K_0(B')$ sending $[1_B]$ to $[1_{B'}]$.

To see the converse, first notice that, for any $B \in \mathcal{L}$, $B = \mathrm{M}_d(L_n)$ for suitable $d, n \in \mathbb{N}$. As noted previously, $(K_0(\mathrm{M}_d(L_n)), [1_{\mathrm{M}_d(L_n)}]) \cong (\mathbb{Z}/(n-1)\mathbb{Z}, \bar{d})$. Hence, if $B' = \mathrm{M}_k(L_m)$ for suitable $k, m \in \mathbb{N}$, then the existence of an isomorphism $\phi : K_0(B) \to K_0(B')$ forces that $n = m$.

Now, since every automorphism of $\mathbb{Z}/(n-1)\mathbb{Z}$ is given by multiplication by an element $1 \leqslant l \leqslant n-1$ such that $\gcd(l, n-1) = 1$, the hypothesis $\phi([1_B]) = [1_{B'}]$ yields that $\bar{k} = \overline{dl} \in \mathbb{Z}/(n-1)\mathbb{Z}$, i.e., that $k \equiv dl \pmod{n-1}$. So Proposition 5.2 gives that

$$\mathrm{M}_k(L_n) \cong \mathrm{M}_{dl}(L_n) \cong \mathrm{M}_d\big(\mathrm{M}_l(L_n)\big).$$

Since g.c.d.$(l, n-1) = 1$, we have $\mathrm{M}_l(L_n) \cong L_n$ by Theorem 5.9. Hence, $\mathrm{M}_d(\mathrm{M}_l(L_n)) \cong \mathrm{M}_d(L_n)$, whence

$$\mathrm{M}_k(L_n) \cong \mathrm{M}_{dl}(L_n) \cong \mathrm{M}_d\big(\mathrm{M}_l(L_n)\big) \cong \mathrm{M}_d(L_n),$$

as desired. □

For reasons identical to those given prior to the statement of Theorems 5.10, 5.11 yields the following result for matrices over Cuntz–Krieger C^*-algebras.

Theorem 5.12. *Let $\mathcal{L}$ denote the set of purely infinite simple C^*-algebras*

$$\big\{\mathrm{M}_d(\mathcal{O}_n) \bigm| d, n \in \mathbb{N}\big\}.$$

Let $B, B' \in \mathcal{L}$. Then $B \cong B'$ if and only if there is an isomorphism $\phi : K_0(B) \to K_0(B')$ for which $\phi([1_B]) = [1_{B'}]$.

We conclude with two remarks. First, we observe that the sequence of isomorphisms between $L(R_n^d)$ and $L(R_n)$ given in the proof of Theorem 5.9 begins with a graph having two vertices, eventually winds its way through graphs having three vertices, and finally concludes with a graph having only one vertex. As it turns out, in any situation for which $t \geqslant d - r - 1$, we are in fact able to establish the isomorphism $L(R_n^d)$ and $L(R_n)$ without utilizing graphs having three vertices. (However, we have been unable to achieve the desired isomorphism in general without using graphs having three vertices.) Second, we see no *a priori* reason why Theorems 2.3 and 2.8 should be expected to contain enough information to provide us with proofs of The Classification Question in the situations described in both Sections 4 and 5. That these two theorems suffice to cover both of these situations suggests that these two theorems may indeed suffice to settle The Classification Question in further generality.

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References

[1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, J. Algebra 293 (2005) 319–334.

[2] G. Abrams, G. Aranda Pino, Purely infinite simple Leavitt path algebras, J. Pure Appl. Algebra 207 (2006) 553–563.

[3] G. Abrams, G. Aranda Pino, Leavitt path algebras of arbitrary graphs, Houston J. Math., in press.

[4] G. Abrams, P.N. Ánh, E. Pardo, Isomorphisms between Leavitt algebras and their matrices, J. Reine Angew. Math., in press.

[5] P. Ara, K. Goodearl, E. Pardo, K_0 of purely infinite simple regular rings, K-Theory 26 (2002) 69–100.

[6] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-Theory for graph algebras, Algebr. Represent. Theory 10 (2007) 157–178.

[7] M. Brustenga, Àlgebres associades a un buirac, PhD thesis, Universitat Autònoma de Barcelona, Spain, 2007.

[8] T. Bates, D. Pask, Flow equivalence of graph algebras, Ergodic Theory Dynam. Systems 24 (2004) 367–382.

[9] E. Kirchberg, The classification of purely infinite C^*-algebras using Kasparov theory, preprint.

[10] W.G. Leavitt, The module type of a ring, Trans. Amer. Math. Soc. 42 (1962) 113–130.

[11] N.C. Phillips, A classification theorem for nuclear purely infinite simple C^*-algebras, Doc. Math. 5 (2000) 49–114.

[12] M.V. Pimsner, A class of C^*-algebras generalizing both Cuntz–Krieger algebras and crossed products by $\mathbb{Z}$, in: Fields Inst. Commun., vol. 12, 1997, pp. 189–212.

[13] I. Raeburn, Graph Algebras, CBMS Reg. Conf. Ser. Math., vol. 103, American Mathematical Society, Providence, ISBN 0-8218-3660-9, 2005.

[14] M. Tomforde, Structure of graph C^*-algebras and generalizations, in: G. Aranda Pino, F. Perera, M. Siles Molina (Eds.), Graph Algebras: Bridging the Gap between Analysis and Algebra, Universidad de Málaga Press, ISBN 978-84-9747-177-0, 2007.

[15] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, J. Algebra 318 (2007) 270–299.