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Type-II hidden symmetries through weak symmetries for nonlinear partial differential equations

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ABSTRACT

The Type-II hidden symmetries are extra symmetries in addition to the inherited symmetries of the differential equations when the number of independent and dependent variables is reduced by a Lie point symmetry. In [B. Abraham-Shrauner, K.S. Govinder, Provenance of Type II hidden symmetries from nonlinear partial differential equations, J. Nonlinear Math. Phys. 13 (2006) 612–622] Abraham-Shrauner and Govinder have analyzed the provenance of this kind of symmetries and they developed two methods for determining the source of these hidden symmetries. The Lie point symmetries of a model equation and the two-dimensional Burgers' equation and their descendants were used to identify the hidden symmetries. In this paper we analyze the connection between one of their methods and the weak symmetries of the partial differential equation in order to determine the source of these hidden symmetries. We have considered the same models presented in [B. Abraham-Shrauner, K.S. Govinder, Provenance of Type II hidden symmetries from nonlinear partial differential equations, J. Nonlinear Math. Phys. 13 (2006) 612–622], as well as the WDVV equations of associativity in two-dimensional topological field theory which reduces, in the case of three fields, to a single third order equation of Monge–Ampère type. We have also studied a second order linear partial differential equation in which the number of independent variables cannot be reduced by using Lie symmetries, however when is reduced by using nonclassical symmetries the reduced partial differential equation gains Lie symmetries.

1. Introduction

Many partial differential equations (PDE's) of physical importance are nonlinear partial differential equations. While there is no existing general theory for solving such equations the methods of point transformations are a powerful tool. One of the most useful point transformations are those which form a continuous group. Lie classical symmetries admitted by nonlinear PDE's are useful for finding invariant solutions. The classical symmetry method for differential equations is based on Lie group symmetries.

If an ordinary differential equation (ODE) is invariant under a Lie group the order of this ODE can be reduced by one, further reductions can be obtained if there is a solvable group. If an ODE loses (gains) a symmetry in addition to the one used to reduce the order of the ODE, the ODE possesses a Type I (Type II) hidden symmetry. Hidden symmetries for ODEs have been extensively studied [1,2] and references there.

If a PDE is invariant under a Lie group, the number of independent variables can be reduced by one. The reduced equation loses the symmetry used to reduce the number of variables and may lose other Lie symmetries depending on the structure of the associated Lie algebra. If a PDE loses (gains) a symmetry in addition to the one used to reduce the number

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of independent variables of the PDE, the PDE possesses a Type I (Type II) hidden symmetry [3]. In fact a Type-II hidden symmetry is a Lie symmetry appearing in the reduced (by means of a classical symmetry) differential equation that was not inherited from the preceding partial differential equation. Unlike the case of ODE's hidden symmetries of PDE's have not been studied extensively. Several examples of Type-II hidden symmetries have been reported in [4,6,10,13]. None of these reports suggested an origin for the Type-II hidden symmetries found by reducing the number of variables of PDE's. It has been noted [4] that these Type-II hidden symmetries do not arise from contact symmetries or nonlocal symmetries as was true for ODEs. This holds since only variable transformations of the PDEs are used. Thus the origin of these hidden symmetries must be in point symmetries [3]. In [3] B. Abraham-Shrauner and K.S. Govinder have identified a common provenance for the Type-II hidden symmetries of differential equations reduced from PDE's that covers the PDE's studied. They pointed out that the crucial point is that the differential equation that is reduced from a PDE and possesses a Type-II hidden symmetry is also a reduced differential equation from one or more other PDE's. The inherited symmetries from these other PDE's are a larger class of Lie point symmetries that includes the Type-II hidden symmetries. The Type-II hidden symmetries are actually inherited symmetries from one or more of the other PDE's. The crucial question [3] is whether we can identify the PDE's from which the Type-II hidden symmetries are inherited. In [3] two methods were proposed: some PDE's may be constructed by calculating the invariants by reverse transformations and some PDE's may be identified by inspection.

The main goal of this paper is to show that the provenance of the Type II Lie point hidden symmetries found for differential equations can also be explained by considering some weak symmetries or conditional symmetries of the original PDE. These *weak symmetries* were introduced in Olver and Rosenau [11]. Their approach consists in calculating the symmetries of the basic equation supplemented by certain differential constraints, chosen in order to weaken the invariance criterion of the basic system and to provide us with the larger Lie-point symmetry groups for the augmented system. Moreover, weak symmetries are derived not only from this overdetermined system but also from all its integrability conditions. In this way one obtains an overdetermined nonlinear system of equations and the solution set is, in this case, quite larger than the corresponding to classical symmetries. In the last years there has been a continuous interest on the topic of nonclassical and weak symmetries. Surveys of these researchers are reported in Olver and Vorobev [12], and Clarkson [5]. In Olver and Rosenau [11] the key question seems to be that *the reductions methods can be unified by the concept of a differential equation with a* ***side condition***. In Saccomandi [14] the key question is which side conditions are admissible providing genuine solutions to the given differential equations and it was shown that weak symmetries are not only of academic interest, but are necessary to recover all the solutions of the Navier–Stokes equations found by the semi-inverse method.

The conditional symmetries can be of two classes: those where the side condition is a Lie point symmetry of the nonlinear PDE alone and those where the side condition is not a Lie point symmetry of the nonlinear PDE alone but of the combined set. In [15] Irina Yehorchenko reported examples of the second class.

In this paper, we focus our attention in weak symmetries of the partial differential equations with special differential constraint in order to determine the source of these Type-II hidden symmetries. The main new result is that we can identify the PDE from which the Type-II hidden symmetries are inherited by using as differential constraint the *side condition* from which the reduction has been derived. We are able to explain why some PDE's derived in [3] by guessing and which reduce to the same ODE do not gain the whole set of Lie symmetries.

In [3] the investigation was confined to hidden symmetries of PDE's for which the number of independent variables is reduced by Lie symmetries. We include an example, appearing in [9], in which the number of independent variables can not be reduced by using Lie symmetries. Nevertheless the number of independent variables is reduced by using nonclassical symmetries and the reduced PDE gains Lie symmetries. That is, a particular case of Type-II hidden symmetries is a Lie symmetry appearing in the reduced (by means of a nonclassical symmetry) differential equation that was not inherited from the preceding partial differential equation.

The significance of these Type-II hidden symmetries is that there may be more symmetries in the subsequent reduced differential equations than can be predicted from the Lie algebra of the original PDE. The general premise of this paper is that increased understanding of Type-II [4] as a part of Lie symmetries is a useful endeavour and may lead to improvements in the solution of differential equations.

2. Weak symmetries for the model equation

We begin by considering the model equation introduced in [3]

$$u_{xxx} + u(u_t + cu_x) = 0 \tag{1}$$

where c is a constant and the subscripts denote differentiation with respect to the variable indicated. Applying the Lie classical method to Eq. (1) leads to a four-parameter Lie group. Associated with this Lie group we have a Lie algebra which can be represented by the following generators [3]:

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = (x + 2ct)\partial_x + 3t\partial_t, \qquad \mathbf{v}_4 = ct\partial_x + t\partial_t + u\partial_u. \tag{2}$$

If we reduce Eq. (1) by using the generator $c\mathbf{v}_1 + \mathbf{v}_2$ we get $u = w(z)$, $z = x - ct$ and the reduced ODE is

$$w_{zzz} = 0 \tag{3}$$

which admits a seven-parameter Lie group. The associated Lie algebra can be represented by the following generators

$$\begin{aligned} &\mathbf{w}_1 = \partial_z, \qquad \mathbf{w}_2 = \partial_w, \qquad \mathbf{w}_3 = z^2\partial_w, \qquad \mathbf{w}_4 = z\partial_z, \\ &\mathbf{w}_5 = z\partial_w, \qquad \mathbf{w}_6 = w\partial_w, \qquad \mathbf{w}_7 = \frac{1}{2}z^2\partial_z + zw\partial_w. \end{aligned} \tag{4}$$

The inherited symmetries are $\mathbf{v}_1 \to \mathbf{w}_1$, $\mathbf{v}_3 \to \mathbf{w}_4$, $\mathbf{v}_4 \to \mathbf{w}_6$, all of which can be inferred by looking at the Lie algebra of (1). The other symmetries are Type II symmetries [3]. Two possible methods have been identified in [3] for finding possible PDE's the symmetries of which are inherited in the transformations $w = u$, $z = x - ct$ in (1). The first method proposed is to *guess* possible PDE's, evaluate their Lie point symmetries and then check if the group generators reduce to (4). Some PDE's that reduce to (3) by using the variables z and w and were proposed, by guessing, in [3] are

$$u_{xxx} = 0, \qquad u_{txx} = 0, \qquad u_{ttx} = 0, \qquad u_{ttt} = 0. \tag{5}$$

We propose to have as differential constraint the side condition from which the reduction has been derived. Then we derive weak symmetries, that is, Lie classical symmetries of the original equation and the side condition.

The PDEs from which the hidden symmetries are inherited are the PDEs obtained by substituting in original PDE the side condition

$$cu_x + u_t = 0, \tag{6}$$

and also the differential consequences. By substituting the side condition we get $u_{xxx} = 0$. If we differentiate the side condition (6) twice with respect to x and then replace the first PDE in (5) by zero we get $u_{txx} = 0$. If we differentiate the side condition (6) once with respect to x and once with respect to t and then replace the second PDE in (5) by zero we get $u_{ttx} = 0$. Finally, if we differentiate the side condition (6) twice with respect to t and then replace the third PDE in (5) by zero we get $u_{ttt} = 0$.

We are going to derive some weak symmetries of the model equation (1), choosing as side condition the differential constraint (6) which is associated to the generator $\mathbf{v}_2 + c\mathbf{v}_1$ that has been used to derive the reduction

$$u = w(z), \quad z = x - ct.$$

Applying Lie classical method to the system (1), (6) we get:

$$\xi = \xi(t, x), \qquad \tau = \tau(t), \qquad \phi = \alpha(x, t)u + \beta(x, t),$$

where $\alpha(x, t) = \xi_x(x, t)u + g_1(t)$ and $\xi(x, t)$ and $\beta(x, t)$ must satisfy $\xi_{xxx} = \beta_{xxx} = 0$. To apply the method in practice we use the MACSYMA package [7]. This yields the following generators

$$\begin{aligned} &\mathbf{u}_1 = f_1(t)\partial_x, \qquad \mathbf{u}_2 = f_2(t)\partial_u, \qquad \mathbf{u}_3 = f_3(t)\partial_t, \qquad \mathbf{u}_4 = f_4(t)x^2\partial_u, \\ &\mathbf{u}_5 = f_5(t)x\partial_x, \qquad \mathbf{u}_6 = f_6(t)x\partial_u, \qquad \mathbf{u}_7 = f_7(t)u\partial_u, \qquad \mathbf{u}_8 = f_8(t)\left(\frac{1}{2}x^2\partial_x + xu\partial_u\right), \end{aligned} \tag{7}$$

with $f_i(t)$, $i = 1, \ldots, 8$, arbitrary functions. However, by appropriate choice of polynomials in t for $f_i(t)$) where (and also taking combinations) the group generators reduce to the seven generators (4). These symmetries (7) are really those of the first PDE in (5), namely $u_{xxx} = 0$ where $u = u(x, t)$, and have been derived in [3]. By interchanging x and t the symmetries of $u_{ttt} = 0$ can also be given by (7).

It was pointed out in [3] that symmetry $\mathbf{w}_7$ is not inherited by the other two equations derived by guessing, namely $u_{xxt} = 0$, $u_{xtt} = 0$. Nevertheless we prove that $\mathbf{w}_7$ is inherited as a weak symmetry of any of Eqs. (5) with the side condition

$$u_t + cu_x = 0. \tag{8}$$

The crucial point is that $\mathbf{u}_7$ is a Lie symmetry of any of Eqs. (5) in which we have substituted the side condition (8), and this equation is precisely $u_{xxx} = 0$ or $u_{ttt} = 0$.

Until now we have assumed that the PDEs are all reduced by using the same variables as the original PDE (1). This does not have to be the case, we now consider the following equation introduced in [3]

$$u_{xxx} + u_{xx}\left(u_x + \frac{t}{x}u_t\right) = 0. \tag{9}$$

The generators of the classical symmetries are

$$\mathbf{v}_1 = x\partial_x, \qquad \mathbf{v}_2 = t\partial_t, \qquad \mathbf{v}_3 = \frac{x}{t}\partial_u, \qquad \mathbf{v}_4 = x\log(t)\partial_x + t\log(t)\partial_t + u\partial_u, \qquad \mathbf{v}_5 = \partial_u. \tag{10}$$

If we reduce Eq. (9) by using $\mathbf{v}_1 + \mathbf{v}_2$ then the new independent variable is $z = \frac{x}{t}$ with the dependent variable unchanged. We propose to have as differential constraint the side condition from which the reduction has been derived. Then we derive Lie classical symmetries of the original equation (9) and the side condition which is

$$xu_x + tu_t = 0. \tag{11}$$

The PDE from which the hidden symmetries are inherited is the PDE obtained by substituting in original PDE (9) the side condition (11). By doing that we get $u_{xxx}=0$. Then one requires that the group transformation leaves invariant the set of solutions of (9) and of the side condition (11) we obtain the Lie generators (7). These symmetries (7) are really those of $u_{xxx}=0$ where $u=u(x,t)$.

The same happens if we consider the following example introduced in [3]

$$u_{xxx}+u_{xx}(xu_{xx}+tu_{tx})=0. \tag{12}$$

The generators of the classical symmetries are

$$\begin{aligned} &\mathbf{v}_1=x\partial_x, \qquad \mathbf{v}_2=t\partial_t, \qquad \mathbf{v}_3=t\partial_x, \\ &\mathbf{v}_4=tx\partial_x+t^2\partial_t+tu\partial_u, \qquad \mathbf{v}_5=x\partial_u, \qquad \mathbf{v}_f=f(t)\partial_u. \end{aligned} \tag{13}$$

If we reduce equation (12) by using $\mathbf{v}_4$ then the new independent variable is $z=\frac{x}{t}$ with the new dependent variable w and $u=tw(z)$. We consider the side condition corresponding to this reduction which is

$$xu_x+tu_t=u. \tag{14}$$

The PDE from which the hidden symmetries are inherited is the PDE obtained by substituting in original PDE (12) some differential consequences of the side condition (14). If we differentiate the side condition (14) once with respect to x, we get

$$xu_{xx}+tu_{tx}=0. \tag{15}$$

By substituting (15) into (12) we get $u_{xxx}=0$. Then one requires that the group transformation leaves invariant the set of solutions of (12) and (15) we obtain the Lie generators (7).

3. Two-dimensional Burgers' equation

In [10] the existence of an extra symmetry besides the inherited symmetries of the two-dimensional Burgers' equation under one symmetry reduction was noted. The two-dimensional Burger's equation is

$$u_t+uu_y-u_{xx}-u_{yy}=0. \tag{16}$$

The Lie group generators of (16) which appeared in [3] are:

$$\mathbf{v}_1=\partial_x, \qquad \mathbf{v}_2=\partial_y, \qquad \mathbf{v}_3=\partial_t, \qquad \mathbf{v}_4=t\partial_y+\partial_u, \qquad \mathbf{v}_5=x\partial_x+2t\partial_t+y\partial_y-u\partial_u. \tag{17}$$

If we reduce equation (16) by using the generator $a\mathbf{v}_1+\mathbf{v}_2$ we get $u=w$ and $z=y-\frac{x}{a}$ the reduced ODE is the one-dimensional Burgers' equation

$$w_t+ww_z-\frac{1+a^2}{a^2}w_{zz}=0, \tag{18}$$

which admits a five-parameter Lie group. The associated Lie algebra can be represented by the following generators

$$\mathbf{w}_1=\partial_z, \qquad \mathbf{w}_2=\partial_t, \qquad \mathbf{w}_3=t\partial_z+\partial_w, \qquad \mathbf{w}_4=2t\partial_t+z\partial_z-w\partial_w, \qquad \mathbf{w}_5=t^2\partial_t+tz\partial_z+(z-tw)\partial_w. \tag{19}$$

The symmetries $\mathbf{w}_i$, $i=1,\dots,4$, are inherited symmetries of the two-dimensional Burgers' equation but $\mathbf{w}_5$ is a Type-II hidden symmetry [3]. The origin of this Type-II hidden symmetry has been considered in [3], in order to determine the other possible PDEs the inherited symmetries of which include all the symmetries in (19) the authors made an educated guess which suggests that a good candidate is

$$u_t+uu_y-u_{yy}\left(\frac{1+a^2}{a^2}\right)=0. \tag{20}$$

In order to determine the other possible PDEs the inherited symmetries of which include all the symmetries in (19) we consider the PDE equation obtained considering the original Burgers equation in $(2+1)$ dimensions and the side condition from which the reduction was derived. This side condition associated to generator $a\mathbf{v}_1+\mathbf{v}_2$ is

$$au_x+u_y=0. \tag{21}$$

Consequently we find (20) and

$$u_t-auu_x-\big(1+a^2\big)u_{xx}=0. \tag{22}$$

Applying the classical method to system (16), (21) we get the following generators:

$$\begin{aligned} &\mathbf{u}_1=F_1(y)\partial_x, \qquad \mathbf{u}_2=F_2(y)\partial_y, \qquad \mathbf{u}_3=F_3(y)\partial_t, \\ &\mathbf{u}_4=F_4(y)(t\partial_y+\partial_u), \qquad \mathbf{u}_5=F_5(y)(y\partial_y+2t\partial_t-u\partial_u), \qquad \mathbf{u}_6=F_6(y)\big(yt\partial_y+t^2\partial_t+(y-tu)\partial_u\big), \end{aligned} \tag{23}$$

which already appeared in [3]. Here the $F_i(y)$, $i = 1, \ldots, 6$, are arbitrary functions, however, by appropriate choice of polynomials in y for $F_i(y)$ (and also taking combinations) the group generators reduce to the five generators (19). The novelty of our result is that we can identify the PDEs (20) and (22), which have been found in [3] by guessing and by the more systematic reverse method, by using as differential constraint the *side condition* from which the reduction has been derived. A crucial point is that in order to determine the other possible PDEs the inherited symmetries of which include all the symmetries in (19) we can consider (20) and (22) plus any differential consequence of the side condition (21). For example the following PDE

$$u_t + uu_x = \frac{1+a^2}{a^2} u_{yy} + D\left(u_{xy} + \frac{u_{yy}}{a}\right)\left(u_{xx} + \frac{2u_{xy}}{a} + \frac{u_{yy}}{a^2}\right) \tag{24}$$

with $D = \text{constant}$ is Eq. (3.11) in Ref. [3] can also be determined by the conditional symmetries method presented here. By substituting the side condition into (16) we get (18) and (22). If we differentiate the side condition (21) once with respect to y and then divide by a we get

$$u_{xy} + \frac{u_{yy}}{a} = 0. \tag{25}$$

If we differentiate the side condition (21) once with respect to x and then divide by a we get

$$u_{xx} + \frac{u_{xy}}{a} = 0. \tag{26}$$

If we add condition (25) divided by a with condition (26) we get

$$u_{xx} + 2\frac{u_{xy}}{a} + \frac{u_{yy}}{a^2} = 0. \tag{27}$$

By adding to (18), (25) multiplied by (27) we get (24).

4. Equations of associativity in two-dimensional topological field theory

In [8] we have considered the WDVV equations of associativity arising in two-dimensional topological field theory, which can be represented, in the simplest nontrivial case, by a single third-order equation of the Monge–Ampère type, the following Ferapontov equation

$$f_{xxx} f_{yyy} - f_{xxy} f_{yyx} - 1 = 0. \tag{28}$$

In [8] it was pointed out that a nice way to obtain an hodograph transformation relating two partial differential equations of the Monge–Ampère type, that is the Ferapontov equation (28) and

$$F_{tyy} - F_{ttt} - F_{tty} F_{yyy} = 0 \tag{29}$$

is to rewrite both PDEs as integrable systems of the so-called hydrodynamic type, allowing them to be mapped by a chain of standard transformations to integrable three-wave systems.

Applying the classical method to Eq. (28) leads to a ten-parameter Lie group. Associated with this Lie group we have a Lie algebra which can be represented by the following generators:

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_y, \quad \mathbf{v}_3 = x\partial_x + \frac{3}{2} f \partial_f, \quad \mathbf{v}_4 = y\partial_y + \frac{3}{2} f \partial_f, \quad \mathbf{v}_5 = xy\partial_f,$$
$$\mathbf{v}_6 = x^2\partial_f, \quad \mathbf{v}_7 = y^2\partial_f, \quad \mathbf{v}_8 = x\partial_f, \quad \mathbf{v}_9 = y\partial_f, \quad \mathbf{v}_{10} = \partial_f.$$

In order to construct the optimal system, following Olver, we first have constructed the commutator table and the adjoint table which shows the separate adjoint actions of each element in $\mathbf{v}_i$, $i = 1, \ldots, 10$, as it acts on all other elements. This construction is done easily by summing the Lie series [8].

The corresponding generators of the optimal system of subalgebras are

$\mathbf{v}_3$,

$\mathbf{v}_4$,

$-a\mathbf{v}_3 + b\mathbf{v}_4$,

$\mathbf{v}_3 - \mathbf{v}_4 + a\mathbf{v}_5 + b\mathbf{v}_{10}$,

$\mathbf{v}_3 + 3\mathbf{v}_4 + a\mathbf{v}_6$,

$\mathbf{v}_3 - 3\mathbf{v}_4 + a\mathbf{v}_9$,

$3\mathbf{v}_3 + \mathbf{v}_4 + a\mathbf{v}_7$,

$$-3\mathbf{v}_3+\mathbf{v}_4+a\mathbf{v}_8,$$
$$a\mathbf{v}_2+b\mathbf{v}_3,$$
$$a\mathbf{v}_1+b\mathbf{v}_4,$$
$$a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_5+d\mathbf{v}_6+e\mathbf{v}_7, \tag{30}$$

where a, b, c, d, e are arbitrary real nonzero constants. Each element of the optimal system defines a reduction of Eq. (28) to an ODE and yield some new explicit solutions.

The reduction with the generator $-a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_5+d\mathbf{v}_6+e\mathbf{v}_7$ leads to

$$z=bx+ay,$$
$$f=-\frac{e}{a}xy^2-\left(\frac{be}{a^2}+\frac{c}{2a}\right)x^2y-\left(\frac{b^2e}{3a^3}+\frac{d}{3a}+\frac{bc}{6a^2}\right)x^3+\varphi(z) \tag{31}$$

where φ satisfies the autonomous and linear ODE

$$\left(2a^3b^2e-2a^5d\right)\varphi'''-\left(4be^2+2ace+a^3\right)=0.$$

Taking into account that $z=bx+ay$, x^3 can be written in terms of y and z and f adopts also the form $f=c_3x^3+c_2x^2y+c_1xy^2+c_0y^3+\varphi(z)$ [8]. Setting $\frac{4be^2+2ace+a^3}{2a^3b^2e-2a^5d}=k$ the ODE can be written as $\varphi'''-k=0$ and admits a seven-parameter Lie group. The associated Lie algebra can be represented by the following generators

$$\mathbf{w}_1=\partial_z,\qquad \mathbf{w}_2=\partial_\varphi,\qquad \mathbf{w}_3=z\partial_z+\left(\varphi+\frac{kz^3}{3}\right)\partial_\varphi,\qquad \mathbf{w}_4=z\partial_\varphi,$$
$$\mathbf{w}_5=\left(\varphi-\frac{kz^3}{6}\right)\partial_\varphi,\qquad \mathbf{w}_6=\frac{1}{2}z^2\partial_\varphi,\qquad \mathbf{w}_7=z^2\partial_z+\left(2z\varphi+\frac{kz^4}{6}\right)\partial_\varphi. \tag{32}$$

The inherited symmetries are $\mathbf{v}_1\to\mathbf{w}_1$, $\mathbf{v}_{10}\to\mathbf{w}_2$, $\mathbf{v}_8\to\mathbf{w}_4$, $\mathbf{v}_6\to\mathbf{w}_6$, all of which can be inferred by looking at the Lie algebra of (1) $\mathbf{w}_3$, $\mathbf{w}_5$ and $\mathbf{w}_7$ are Type-II hidden symmetries. We now consider weak symmetries of (28) with the following side condition

$$-af_x+bf_y=cxy+dx^2+ey^2 \tag{33}$$

corresponding to the generator $a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_5+d\mathbf{v}_6+e\mathbf{v}_7$. Applying the classical method to system (28), (33) we get:

$$\mathbf{u}_1=f_1(y)\partial_x,\qquad \mathbf{u}_2=f_2(y)\partial_f,\qquad \mathbf{u}_3=f_3(y)\left(x\partial_x+\left(f+2Mx^3\right)\right)\partial_f,\qquad \mathbf{u}_4=f_4(y)x\partial_f,$$
$$\mathbf{u}_5=f_5(y)\left(f-Mx^3\right)\partial_f,\qquad \mathbf{u}_6=\frac{1}{2}x^2f_6(y)\partial_f,\qquad \mathbf{u}_7=f_7(f)\left(x^2\partial_x+\left(2xf+Mx^4\right)\right)\partial_u, \tag{34}$$

with $f_i(y), i=1,\dots,7$, arbitrary functions and $M=\frac{4ad^2+2bcd+b^3}{2(b^2e-a^2d)}$. However, by appropriate choice of polynomials in y for $f_i(y)$ the group generators reduce to the seven generators (32).

The PDEs the inherited symmetries of which include all the symmetries in (32) are

$$f_{xxx}=M,\qquad f_{yyy}=N, \tag{35}$$

where $N=\frac{4be^2+2ace+a^3}{2(b^2e-a^2d)}$ for $b\neq 0$ and $b^2e-a^2d\neq 0$. These two equations can be easily derived by substituting some differential consequences of the side condition (33) into (28). From (33) $f_y=\frac{1}{b}(af_x+cxy+dx^2+ey^2)$ differentiating this expression twice with respect to x, differentiating this twice with respect to x and y, differentiating twice with respect to y and substituting into (28) we get, for $b\neq 0$ and $b^2e-a^2d\neq 0$, the first equation in (35). In a similar way we get the second one.

5. Second order linear PDE

In [9] a new procedure was introduced for which PDEs allowing no classical symmetry groups can indeed be solved using symmetries. The method entails expanding the dimensionality using a simple group to a higher dimensional PDE which then allows multiple group reductions to obtain particular solutions. The authors pointed out that for non-Abelian two parameter group and the corresponding two-parameter algebra, spanned by v_1 and v_2 which can always be written $[v_1, v_2]$=v_2 by suitable choice of basis, it is necessary to use first generator v_2 in order for the second reduction to be allowed. The following possibility was pointed out in [9] for the following second order linear PDE with various coefficients depending on the independent variables x and y:

$$\left(1+x^2\right)F_{xx}+4\frac{y}{x}\left(F_{xy}+\frac{y}{x}F_{yy}\right)+\left(2x+\frac{1}{x}\right)F_x+\frac{y}{x^2}(ay+4)F_y=0. \tag{36}$$

This equation has two point symmetry invariance groups, whose generators are $\mathbf{v}=\partial_F$ and $\mathbf{v}=F\partial_F$ neither of these symmetries can be used to reduce the number of independent variables. In [9], the authors have chosen a simple scaling group to enlarge the dimensionality of (36). This fact suggest tu choose a simple scaling group as a side condition. We observe that Eq. (36) admits the conditional symmetry generator

$$\mathbf{v}=x\partial_x+2y\partial_y. \tag{37}$$

If we reduce equation (36) by using the generator (37) we get $w=F$ and $z=\frac{x}{\sqrt{y}}$ the reduced ODE is

$$z^2w_{zz}+\left(2z-\frac{a}{2z}\right)w_z=0, \tag{38}$$

which admits an eight-parameter Lie group. The associated Lie algebra can be represented by the following generators

$$\begin{aligned}
&\mathbf{w}_1=\partial_w, \qquad \mathbf{w}_2=\frac{2\sqrt{\pi}z^2}{\sqrt{a}}E_1e^{\frac{a}{4z^2}}\partial_z,\\
&\mathbf{w}_3=w\partial_w, \qquad \mathbf{w}_4=\frac{2\sqrt{\pi}z^2}{\sqrt{a}}E_1we^{\frac{a}{4z^2}}\partial_z-2\frac{|z|w^2}{z}\partial_w,\\
&\mathbf{w}_5=z^2e^{\frac{a}{4z^2}}\partial_z, \qquad \mathbf{w}_6=\left(-\frac{\sqrt{\pi}z^2}{2\sqrt{a}}e^{\frac{a}{4z^2}}G+\frac{2\pi s(z)z^2EE_1e^{\frac{a}{4z^2}}}{a}\right)\partial_z-\frac{\sqrt{\pi}}{a}Ew\partial_w,\\
&\mathbf{w}_7=z^2we^{\frac{a}{4z^2}}\partial_w, \qquad \mathbf{w}_8=-\frac{\sqrt{\pi}E}{\sqrt{a}}\partial_w,
\end{aligned} \tag{39}$$

with $k_i, i=1,\dots,8,$ arbitrary constants and

$$E_1=\operatorname{erf}\left(\frac{\sqrt{a}}{2|z|}\right), \qquad E=\operatorname{erf}\left(\frac{\sqrt{a}}{2z}\right), \qquad G=\int\frac{4E_1e^{-\frac{a}{4z^2}}}{z|z|}dz, \quad s(z)=\operatorname{signum}(z).$$

The inherited symmetries are $\mathbf{v}_1\to\mathbf{w}_1$, $\mathbf{v}_3\to\mathbf{w}_3$, all of which can be inferred by looking at the Lie algebra of (36). The other symmetries are special Type II symmetries. These hidden symmetries are extra Lie symmetries that appear when the number of variables of a PDE is reduced by a variable transformation found from a nonclassical symmetry of the PDE.

We derive the symmetries of the basic equation supplemented by the following differential constraint

$$xF_x+2yF_y=0, \tag{40}$$

which correspond to the nonclassical generators (37). Applying the classical method to the system (36), (40) we get:

$$\xi=\alpha(x,y)F+\beta(x,y), \qquad \tau=\tau(y), \qquad \phi=\left[\alpha\left(\frac{ay}{2x^3}-\frac{2}{x}\right)+\alpha_x\right]F^2+\gamma(x,y)F+\delta(x,y).$$

This yields

$$\begin{aligned}
&\mathbf{u}_1=g_1(y)\partial_F, \qquad \mathbf{u}_2=\frac{2\sqrt{\pi}x^2g_2(y)}{\sqrt{ay}}E_1e^{\frac{ay}{4x^2}}\partial_x,\\
&\mathbf{u}_3=g_3(y)F\partial_F, \qquad \mathbf{u}_4=g_4(y)\left(\frac{2\sqrt{\pi}x^2}{\sqrt{ay}}E_1Fe^{\frac{ay}{4x^2}}\partial_x-2\frac{|x|F^2}{x}\partial_F\right),\\
&\mathbf{u}_5=g_5(y)x^2e^{\frac{ay}{4x^2}}\partial_x, \qquad \mathbf{u}_6=g_6(y)\left(\left(-\frac{\sqrt{\pi}x^2}{2\sqrt{ay}}e^{\frac{ay}{4x^2}}G+2\pi s(x)EE_1\frac{x^2e^{\frac{ay}{4x^2}}}{ay}\right)\partial_x-\frac{\sqrt{\pi}}{\sqrt{ay}}EF\partial_F\right),\\
&\mathbf{u}_7=g_7(y)x^2Fe^{\frac{ay}{4x^2}}\partial_F, \qquad \mathbf{u}_8=-\frac{\sqrt{\pi}g_8(y)E}{\sqrt{ay}}\partial_F,
\end{aligned} \tag{41}$$

$$\mathbf{u}_\tau=-\left(\frac{\sqrt{a\pi}E_1|x|e^{\frac{ay}{4x^2}}}{4x\sqrt{y}}+\frac{\sqrt{\pi}x^2}{2\sqrt{ay}}e^{\frac{ay}{4x^2}}M\right)\tau(y)\partial_x, \tag{42}$$

with

$$E_1=\operatorname{erf}\left(\frac{\sqrt{ay}}{2|x|}\right), \qquad E=\operatorname{erf}\left(\frac{\sqrt{ay}}{2x}\right), \qquad G=\int\frac{4E_1e^{-\frac{ay}{4x^2}}}{x|x|}dx, \qquad M=\int\frac{aE_1}{x^2|x|}dx, \qquad s(x)=\operatorname{signum}(x).$$

The functions $g_i(y), i=1,\dots,8,$ and $\tau=\tau(y)$ are arbitrary functions. However, by appropriate choice of polynomials in y for $g_i(y)$ (and also taking combinations) the group generators reduce to the eight generators (39). The PDEs from which for the inherited symmetries became the Type-II hidden symmetries can be found from (36), (40) and two PDEs found as differential consequences from (40) by eliminating derivatives of F. These PDEs the inherited symmetries of which include all the symmetries in (36) are

$$x^2 F_{xx} + F_x\left(2x - \frac{ay}{2x}\right) = 0, \tag{43}$$

$$F_{yy} + \left(\frac{a}{4x^2} + \frac{1}{2y}\right)F_y = 0. \tag{44}$$

We remark that, as far as we know, this is the first time in which this kind of Type-II hidden symmetries have been found. The original equation admits a two-parameter Lie group while the reduced (by means of a nonclassical symmetry) ODE admits an eight-parameter Lie group so we have six Type-II hidden symmetries. In order to find the PDEs from which these hidden symmetries arise we have considered the basic equation supplemented by the differential constraint or side condition (40) corresponding to the nonclassical generators (37). The crucial point is that generators $\mathbf{u}_i$, with $i = 3, \ldots, 8$, are Lie symmetries of any of Eqs. (43) and (44) that have been derived by substituting into Eq. (36) the side condition (40).

6. Conclusions

The Type-II hidden symmetries are extra symmetries that appear when the number of variables of a PDE is reduced by a variable transformation found from a Lie symmetry of the PDE. In [3] two methods were presented for finding one or more PDEs from which the Type-II hidden symmetries are inherited. We have analyzed the connection between one of these methods and weak symmetries of the PDE with special differential constraint in order to determine the source of the Type-II hidden symmetries. We have considered the same models presented in [3], as well as the WDVV equations of associativity in two-dimension topological field theory which reduces, in the case of three fields, to a single third order equation of Monge–Ampère type. In [3] the investigation was confined to hidden symmetries of PDEs for which the number of independent variables is reduced by Lie symmetries. We include an example, appearing in [9], in which the number of independent variables can not be reduced by using Lie symmetries. Nevertheless the number of independent variables is reduced by using nonclassical symmetries and the reduced PDE gains Lie symmetries. The novelty of our result is that we can identify the PDEs, which has been found in [3] by guessing, by using as differential constraint the *side condition* from which the reduction has been derived. The significance of these Type-II hidden symmetries is that there may be more symmetries in the subsequent reduced differential equations than can be predicted from the Lie algebra of the original PDE. The general premise of this paper is, that increased understanding of Type-II hidden symmetries [4] as a part of Lie symmetries is a useful endeavor and may lead to improvements in the solution of differential equations.

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