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A characterization of pseudoinvexity in multiobjective programming

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ABSTRACT

In this paper, we introduce new classes of vector functions which generalize the class of scalar invex functions. We prove that these new classes of vector functions are characterized in such a way that every vector critical point is an efficient solution of a Multiobjective Programming Problem. We establish relationships between these new classes of functions and others used in the study of efficient and weakly efficient solutions.

1. Introduction

Mathematical Programming requires the study of optimality conditions and the properties of the functions that are involved in the problems. In this sense, it is well known that convex functions play an important role in Mathematical Programming because many important results require the restrictive assumption of convexity. But in general, convexity is not a necessary condition. Generalized Convexity is an attempt to find types of functions that are similar to convex functions, in the sense that they share some of their desirable properties.

We will centre on Multiobjective Programming Problems, which, in general, can be formulated as follows

$$(MP) \quad \text{Minimize } f(x) = (f_1(x), \dots, f_p(x)) \quad \text{subject to: } x \in S \subseteq \mathbb{R}^n$$

where $f : S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ and S is said to be the feasible set.

We introduce the following convention for equalities and inequalities.

If $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, then

$$x = y \Leftrightarrow x_i = y_i, \quad \forall i = 1, \dots, n,$$
$$x < y \Leftrightarrow x_i < y_i, \quad \forall i = 1, \dots, n,$$
$$x \leqq y \Leftrightarrow x_i \leq y_i, \quad \forall i = 1, \dots, n,$$
$$x \leq y \Leftrightarrow x_i \leq y_i, \quad \forall i = 1, \dots, n, \quad \text{and there exists } j \text{ such that } x_j < y_j.$$

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We denote $\vec{0} = (0, 0, \ldots, 0) \in \mathbb{R}^k$, for $k = 1, 2, \ldots$. And to avoid confusion, we use 0 instead of $\vec{0}$ when $k = 1$. The solutions of (MP) are named efficient points.

Definition 1. A feasible point, $\bar{x}$, is said to be an efficient solution of (MP) if there does not exist another feasible point, x, such that $f(x) \leq f(\bar{x})$.

Later, a more general concept appeared such as the weakly efficient solution of (MP) (replace $f(x) \leq f(\bar{x})$ by $f(x) < f(\bar{x})$ in the definition above). In order to establish the optimality condition we will employ the following definition. Let $\mathbb{M}^{p\times n}$ denote the set of $p \times n$ real matrices.

Definition 2. A feasible point for (MP), $\bar{x}$, is said to be a vector critical point if there exists $\lambda \in \mathbb{R}^p$, such that

$$\lambda \nabla f(\bar{x}) = \vec{0},$$
$$\lambda \geq \vec{0},$$

where $\nabla f(\bar{x}) \in \mathbb{M}^{p\times n}$ is the gradient matrix of the vector function f.

The vector critical point condition is necessary for a feasible point of (MP) to be an efficient solution or a weakly efficient solution (see [1,2]). It is an interesting problem to look for the class of functions for which the converse holds. Several kinds of functions have appeared in the literature in reply to this question. The introduction of the class of invex functions for the scalar case ($p = 1$) by Craven and Hanson (see [3,4]) closed the scalar problem, in the sense that it is the more general class of functions satisfying the converse. Hanson [4] noted that there are simple extensions of invex functions, the pseudoinvex functions. However, in the scalar case, both of these kinds of functions are equivalent (see Ben-Israel and Mond [5]).

Early, researchers have extended these results to multiobjective problems. So, Osuna et al. [6,7] have proposed a new kind of vector functions (it is called pseudoinvexity-I, in this paper) and they have proved that it is necessary and sufficient for the set of vector critical points and the set of weakly efficient solutions of (MP) to be the same.

Hanson [8] have introduced vector type invexity extending the pseudo, quasi, quasi-pseudo, pseudo-quasi type-I invexity of Kaul et al. [9], obtaining sufficiency results for efficient solutions.

In the present paper, we introduce the pseudoinvex-II functions and we extend the study of Osuna et al. [6,7] to provide necessity and sufficiency results for efficient solutions of (MP) under pseudoinvexity-II; and in this manner, we offer improvement on the conditions of Hanson [8], what is treated in Section 2. In Section 3, we study the existing relationships among vector invex, pseudoinvex-I and pseudoinvex-II functions, by some examples, and we prove that these classes of vector functions are different, from what happened in scalar case.

2. Efficiency. Characterization

The following definition extends the concept of scalar invex function to the multiple case (see [6,7]).

Definition 3. Let $f = (f_1, \ldots, f_p) : S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open set S. Then, the vector function f is said to be invex if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall x, \bar{x} \in S$

$$f(x) - f(\bar{x}) \geqq \nabla f(\bar{x})\eta(x, \bar{x})^T.$$

The next proposition characterizes the invex vectorial function.

Proposition 1. *Let $f = (f_1, \ldots, f_p) : S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open set S. Then f is invex with respect to the vector function η if and only if the scalar functions f_i, $1 \leq i \leq p$, are invex with respect to the same vector function η.*

The proof of Proposition 1 is immediate from Definition 3. Next, we define two classes of functions generalizing the class of scalar pseudoinvex functions. We call them pseudoinvex-I and pseudoinvex-II.

Definition 4. Let $f = (f_1, \ldots, f_p) : S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open set S. Then the vector function f is said to be pseudoinvex-I if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall x, \bar{x} \in S$

$$f(x) - f(\bar{x}) < \vec{0} \Rightarrow \nabla f(\bar{x})\eta(x, \bar{x})^T < \vec{0}.$$

Definition 5. Let $f = (f_1, \ldots, f_p) : S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open set S. Then the vector function f is said to be pseudoinvex-II if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall x, \bar{x} \in S$

$$f(x) - f(\bar{x}) \leq \vec{0} \Rightarrow \nabla f(\bar{x})\eta(x, \bar{x})^T < \vec{0}.$$

In the scalar case, i.e, when $p = 1$, Definitions 3–5 are equivalent (see [5]). Osuna et al. [6], have proved that pseudoinvexity-I is necessary and sufficient for the set of vector critical points and the set of weakly efficient solutions of (MP) to be the same.

Theorem 1. *Every vector critical point is a weakly efficient solution of (MP) if and only if f is pseudoinvex-I.*

This result generalizes to the multiple case the characterization result for scalar invex function. We will prove that for the set of vector critical points and the set of efficient solutions of (MP) to coincide it is necessary and sufficient for the class of functions to be the class of pseudoinvex-II functions. With this aim, we give a sufficient condition.

Theorem 2. *Let $\bar{x}$ be a feasible point for (MP) and let f be a pseudoinvex-II function. If $\bar{x}$ is a vector critical point, then $\bar{x}$ is an efficient solution of (MP).*

Proof. Let us suppose that $\bar{x}$ is a vector critical point. We have to prove that $\bar{x}$ is an efficient solution for (MP). Let us suppose that this statement is false. Consequently, there exists a feasible point x such that

$$f(x) - f(\bar{x}) \leq \vec{0}.$$

As f is pseudoinvex-II, it follows that there exists η such that

$$\nabla f(\bar{x})\eta(x,\bar{x})^T < \vec{0}. \tag{1}$$

Since $\bar{x}$ is a vector critical point, $\exists\bar{\lambda} \geq \vec{0}$ such that

$$\bar{\lambda}\nabla f(\bar{x}) = \vec{0},$$

and taking dot product with $\eta(x,\bar{x})$, we obtain

$$\bar{\lambda}\nabla f(\bar{x})\eta(x,\bar{x})^T = 0. \tag{2}$$

As $\bar{\lambda} \geq \vec{0}$, from (1) we have

$$\bar{\lambda}\nabla f(\bar{x})\eta(x,\bar{x})^T < 0,$$

which contradicts (2), and hence, $\bar{x}$ is an efficient solution of (MP). □

But pseudoinvexity-II is, moreover, a necessary condition. This is established in the next theorem.

Theorem 3. *If every vector critical point is an efficient solution of (MP), then f is pseudoinvex-II.*

Proof. We begin by supposing that there exist two feasible points x and $\bar{x}$ such that

$$f(x) - f(\bar{x}) \leq \vec{0}.$$

Consequently, $\bar{x}$ is not an efficient solution, and hence by hypothesis, $\bar{x}$ is not a vector critical point, i.e.,

$$\bar{\lambda}\nabla f(\bar{x}) = \vec{0}$$

has no solution $\bar{\lambda} \geq \vec{0}$. Let $U(\bar{x})$ be a subset, $U(\bar{x}) \subseteq \mathbb{R}^n$, such that $u \in U(\bar{x})$ if and only if it is fulfilled

$$\nabla f(\bar{x})u^T < \vec{0}.$$

By Gordan's Theorem [10], it follows that $U(\bar{x})$ is nonempty. To prove that f is pseudoinvex-II it is enough to find $\bar{\eta}(x,\bar{x}) \in \mathbb{R}^n$ such that Definition 5 is satisfied. Since $U(\bar{x}) \neq \emptyset$, choice $\bar{u} \in U(\bar{x})$, and define $\bar{\eta}(x,\bar{x}) = \bar{u}$. Hence, if $f(x) - f(\bar{x}) \leq \vec{0}$ is verified, there exists a vector function $\bar{\eta}$ such that

$$f(\bar{x})\bar{\eta}(x,\bar{x})^T < \vec{0}$$

and we conclude that f is pseudoinvex-II. □

Summarizing, the following theorem states the main result in this section.

Theorem 4. *Every critical point is an efficient solution of (MP) if and only if f is pseudoinvex-II.*

Therefore, the equivalence between vector critical points and efficient solutions of (MP) characterizes the class of pseudoinvex-II vector functions. This theorem generalizes scalar characterization of invex functions to the multiobjective case.

3. Relationships between the classes of vector functions

In the scalar case, we know that the classes of invex functions, pseudoinvex-I functions and pseudoinvex-II functions are equivalent. However, we will show that this equivalence is not true in the vectorial case, not even between any two of these types of functions. Now, we will establish the relationships between these classes of vector functions.

Let us begin with invexity and pseudoinvexity-I. From Definitions 3 and 4, we have that if f is invex then f is pseudoinvex-I. The converse is not true as the next example shows.

Example 1 (f *Pseudoinvex-I* $\not\Rightarrow$ f *Invex*). Let us consider $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$, with $f(x) = (f_1(x), f_2(x)) = (x^2, -x^2)$.$\forall$: $x, \bar{x} \in \mathbb{R}$ we have that $x^2 - \bar{x}^2 < 0 \Leftrightarrow -x^2 + \bar{x}^2 > 0$, i.e.,

$$f_1(x) - f_1(\bar{x}) < 0 \Leftrightarrow f_2(x) - f_2(\bar{x}) > 0.$$

Hence, $\nexists x, \bar{x} \in \mathbb{R}$ such that $f_i(x) - f_i(\bar{x}) < 0, \quad i = 1, 2$, which implies that $f(x) - f(\bar{x}) < \vec{0}$ is not verified, and therefore f is pseudoinvex-I with respect to any function η.

On the other hand, f_2 is not invex because $\nabla f_2(0) = 0$ and $\bar{x} = 0$ is not a minimum for this function. From Proposition 1, we conclude that f is not invex.

Examples 2 and 3 show that the classes of invex functions and pseudoinvex-II functions are different.

Example 2 (f *Invex* $\not\Rightarrow$ f *Pseudoinvex-II*). Let us consider $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$, with $f(x) = (f_1(x), f_2(x)) = (x^2, 5)$. We have that $\nabla f(x) = (2x, 0)$. $\forall x, \bar{x} \in \mathbb{R}$, we first prove that f is invex for the vector

$$\eta(x, \bar{x}) = \begin{cases} \dfrac{x^2 - \bar{x}^2}{2\bar{x}} & \text{if } \bar{x} \neq 0, \\ 1 & \text{if } \bar{x} = 0. \end{cases}$$

It follows that

$$f(x) - f(\bar{x}) = (x^2 - \bar{x}^2, 0) = \left\{ \begin{matrix} \left(\dfrac{x^2 - \bar{x}^2}{2\bar{x}} \cdot 2\bar{x}, 0 \right) & \text{if } \bar{x} \neq 0 \\ (x^2, 0) & \text{if } \bar{x} = 0 \end{matrix} \right\} \geqq \nabla f(\bar{x}) \eta(x, \bar{x}).$$

In consequence, f is invex.

On the other hand, f is not pseudoinvex-II. By choosing $x = 1, \bar{x} = 2$, we have

$$f(x) - f(\bar{x}) = (-3, 0) \leq \vec{0}.$$

Since $\nabla f(\bar{x}) = (4, 0)$, it follows that $\nabla f(\bar{x})u = (4, 0)u = (4u, 0) \not< \vec{0}$, $\forall u \in \mathbb{R}$. Hence, there does not exist a function η such that $\nabla f(\bar{x})\eta(x, \bar{x}) < \vec{0}$, and in consequence f is not pseudoinvex-II.

Example 3 (f *Pseudoinvex-II* $\not\Rightarrow$ f *Invex*). Let us consider $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$, with $f(x) = (f_1(x), f_2(x)) = (x^2, -x^2)$. From Example 1 we know that f is not invex.

We now prove that f is pseudoinvex-II. We have that $f(x) - f(\bar{x}) = (x^2 - \bar{x}^2, \bar{x}^2 - x^2) \leq \vec{0} \iff$

$$x^2 - \bar{x}^2 < 0, \quad \text{and} \quad \bar{x}^2 - x^2 \leq 0 \tag{3}$$

or

$$x^2 - \bar{x}^2 \leq 0, \quad \text{and} \quad \bar{x}^2 - x^2 < 0. \tag{4}$$

If $x^2 - \bar{x}^2 < 0$, then $\bar{x}^2 - x^2 > 0$ and (3) is not verified. By a similar argument, (4) is not verified. Therefore, the inequality $f(x) - f(\bar{x}) \leq \vec{0}$ is not verified, and we conclude that f is pseudoinvex-II.

The intersection between invex functions set and pseudoinvex-II functions set is a nonempty set, since a linear function is invex, pseudoinvex-I and pseudoinvex-II.

Let us continue our study of pseudoinvex-I and pseudoinvex-II functions. From Definitions 4 and 5 it is easy to see that the class of pseudoinvex-I functions contains the class of pseudoinvex-II functions. The converse is not true, as it is shown in Example 4.

Example 4 (f *Pseudoinvex-I* $\not\Rightarrow$ f *Pseudoinvex-II*). Let us consider $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$, with $f(x) = (f_1(x), f_2(x)) = (x^2, 5)$. From Example 2 we know that f is invex and is not pseudoinvex-II. Besides, as f is invex, it follows that f is pseudoinvex-I.

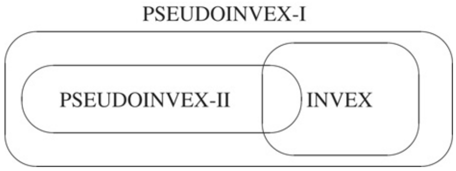

Fig. 1.

Amongst the consequences derived from these results, let us emphasize that the class of pseudoinvex-I functions contains the class of invex functions and the class of pseudoinvex-II. In order to conclude the study of the relationships between these classes of functions, we will prove that this inclusion is strict. Let

PSI $= \{f : S \subseteq R^n \to R^p / f$ is pseudoinvex-I$\}$,

PSII $= \{f : S \subseteq R^n \to R^p / f$ is pseudoinvex-II$\}$,

INV $= \{f : S \subseteq R^n \to R^p / f$ is invex$\}$.

We have the following immediate consequence of definitions before.

Theorem 5. INV $\cup$ PSII $\subset$ PSI.

The above inclusion is strict and $INV \cup PSII \neq PSI$. To show this, the next example give a pseudoinvex-I function which is neither invex nor pseudoinvex-II.

Example 5 ($INV \cup PSII \neq PSI$). Let us consider the function $f : \mathbb{R} \to \mathbb{R}^2$, with $f(x) = (f_1(x), f_2(x)) = (-x^2, 5)$.
We have proved in Example 1 that $f_1(x) = -x^2$ is not invex and by Proposition 1, f is not invex.
f is not pseudoinvex-II. To see this, let $x = 2$ and $\bar{x} = 1$. $f(x) - f(\bar{x}) = (-3, 0) \leq \vec{0}$. However, $\nabla f(\bar{x})u = \nabla f(1)u = (-3u, 0) \nless \vec{0}$, $\forall u \in \mathbb{R}$. In consequence, there exists no $\eta(x, \bar{x})^T$ such that $\nabla f(\bar{x})\eta(x, \bar{x}) < \vec{0}$.
f is pseudoinvex-I, because $\forall x, \bar{x}$ we have $f(x) - f(\bar{x}) = (\bar{x}^2 - x^2, 0) \nless \vec{0}$ and therefore f is pseudoinvex-I.

Consequently, the relationship between invex, pseudoinvex-I and pseudoinvex-II functions is as given in Fig. 1.

4. Conclusions

The type of scalar invex function introduced by Hanson [4], is a necessary and sufficient condition for a critical point to be an optimal solution of a scalar programming problem. We have introduced new classes of functions to generalize this result to the multiobjective problem (MP). In this way, Osuna et al. [6,7] established that a pseudoinvex-I function is characterized in such a way that every critical point is a weakly efficient solution of (MP). We have extended this result to efficient solutions and we have proved that the equivalence between critical points and efficient solutions characterizes the class of pseudoinvex-II functions, which provide improvements from earlier works ([8,9]).

In the scalar case, invex and pseudoinvex functions are equivalent. Vector invex, pseudoinvex-I and pseudoinvex-II functions generalize scalar invex and pseudoinvex functions, but unlike those we have shown that they are not equivalent. Moreover, we have proved that the class of vector invex functions and pseudoinvex-II functions are not equivalent, and that they are strictly contained in the class of pseudoinvex-I functions (Fig. 1).

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