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A SHORT NOTE ABOUT EXPOSED POINTS IN REAL BANACH SPACES*

Antonio Aizpuru
Departamento de Matemáticas, Universidad de Cádiz, Puerto Real, Cádiz, 11510, Spain
Francisco J García-Pacheco
Department of Mathematical Sciences, Kent State University, Kent, Ohio, 44242, USA
E-mail: fgarcia@math.kent.edu

Abstract We express the set of exposed points in terms of rotund points and non-smooth points. As long as we have Banach spaces each time "bigger", we consider sets of non-smooth points each time "smaller".

Key words Exposed point, exposed face, rotund point, smooth point

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1 Background

Let us begin with recalling some geometrical definitions and notions for real Banach spaces. We refer the reader to [1] for a wider perspective of the following concepts. A non-empty subset C of the unit sphere $\mathbf{S}_X$ of a real Banach space X is said to be an exposed face of $\mathbf{B}_X$ if there exists $f \in \mathbf{S}_{X^*}$ such that $C = f^{-1}(1) \cap \mathbf{B}_X$. On the other hand, a point x of $\mathbf{S}_X$ is said to be

(1) an exposed point of $\mathbf{B}_X$ if $\{x\}$ is an exposed face of $\mathbf{B}_X$;

(2) a rotund point of $\mathbf{B}_X$ if every $y \in \mathbf{S}_X$ with $\|(x+y)/2\| = 1$ verifies that $x = y$;

(3) a smooth point of $\mathbf{B}_X$ if every $f, g \in \mathbf{S}_{X^*}$ with $f(x) = g(x) = 1$ verify that $f = g$.

The sets of exposed points of $\mathbf{B}_X$, rotund points of $\mathbf{B}_X$, and smooth points of $\mathbf{B}_X$ will be denoted, respectively, by $\exp(\mathbf{B}_X)$, $\mathrm{rot}(\mathbf{B}_X)$, and $\mathrm{smo}(\mathbf{B}_X)$. It is well known that every rotund point is an exposed point. Furthermore, every exposed point is a rotund point if it is smooth. In this article everything starts with the following fact for a real Banach space X:

$$\begin{aligned}\exp(\mathbf{B}_X) &= [\exp(\mathbf{B}_X) \cap \mathrm{smo}(\mathbf{B}_X)] \cup [\exp(\mathbf{B}_X) \cap \mathbf{S}_X \setminus \mathrm{smo}(\mathbf{B}_X)] \\ &= \mathrm{rot}(\mathbf{B}_X) \cup [\exp(\mathbf{B}_X) \cap \mathbf{S}_X \setminus \mathrm{smo}(\mathbf{B}_X)].\end{aligned}$$

The purpose of this article is to find a set of non-smooth points whose expression does not involve exposed points and so that previous equality keeps valid. We will see that, as long as we have Banach spaces each time "bigger", we will have to consider sets of non-smooth points each time "smaller".

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2 Results

We will begin this section with real Banach spaces of dimension 2. For these spaces, the set of non-smooth points we are looking for is no other than the "biggest" possible one, that is, $\mathbf{S}_X \setminus \mathrm{smo}(\mathbf{B}_X)$, also denoted by $\mathrm{nsmo}(\mathbf{B}_X)$.

Theorem 2.1 Let X be a real Banach space. Then, $\exp(\mathbf{B}_X) \subseteq \mathrm{rot}(\mathbf{B}_X) \cup \mathrm{nsmo}(\mathbf{B}_X)$. If X has dimension 2, then $\exp(\mathbf{B}_X) = \mathrm{rot}(\mathbf{B}_X) \cup \mathrm{nsmo}(\mathbf{B}_X)$.

Proof We already know that $\exp(\mathbf{B}_X) \subseteq \mathrm{rot}(\mathbf{B}_X) \cup \mathrm{nsmo}(\mathbf{B}_X)$. Assume then that X has dimension 2. If $x \in \mathrm{nsmo}(\mathbf{B}_X)$, then it is well known that there are uncountably many functionals in $\mathbf{S}_{X^*}$ attaining their norms at x. Suppose that none of them attains its norm only at x. Then, for each f of them there is $y_f \in \mathbf{S}_X \setminus \{x\}$ so that $f(y_f) = f(x) = 1$. Note that the segments (y_f, x) are pairwise disjoint, therefore we conclude the existence of uncountably many open pairwise disjoint sets in $\mathbf{S}_X$. This fact contradicts the separability of the unit sphere.

Unfortunately, Theorem 2.1 does not hold even in real three dimensions. Indeed, $(1, 1, 0)$ is neither a smooth point nor an exposed point of the unit ball of ℓ_∞^3. As a consequence, we need to consider a "smaller" set of non-smooth points, for which we will introduce a geometrical concept stronger than non-smoothness. It is the time for separable spaces.

Definition 2.2 Let X be a real Banach space. A point $x \in \mathbf{S}_X$ will be said to be a strongly non-smooth point of $\mathbf{B}_X$ if, for every $y \in \mathbf{S}_X \setminus \{x\}$ with $[x, y] \subseteq \mathbf{S}_X$, x is not a smooth point of $\mathbf{B}_Y$, where $Y = \mathrm{span}\{x, y\}$. The set of strongly non-smooth points of $\mathbf{B}_X$ will be denoted by $\mathrm{nsmo}_s(\mathbf{B}_X)$.

Notice that $\mathrm{nsmo}_s(\mathbf{B}_X) \subseteq \mathrm{nsmo}(\mathbf{B}_X)$, in other words, this time the set of non-smooth points is "smaller".

Lemma 2.3 Let X be a real Banach space and consider a point $x \in \mathbf{S}_X$. The following statements are equivalent:

(1) $\{x\} = \bigcap\{C \subseteq \mathbf{S}_X : C$ is an exposed face of $\mathbf{B}_X$ and $x \in C\}$.

(2) x is either a rotund point of $\mathbf{B}_X$ or a strongly non-smooth point of $\mathbf{B}_X$.

(3) For every 2-dimensional subspace Y containing x, x is an exposed point of $\mathbf{B}_Y$.

Proof In the first place, assume that $\{x\}$ is the intersection of all exposed faces of $\mathbf{B}_X$ containing x. Suppose also that x is not a rotund point of $\mathbf{B}_X$. Take any $y \in \mathbf{S}_X \setminus \{x\}$ with $[x, y] \subseteq \mathbf{S}_X$. Let us denote $Y = \mathrm{span}\{x, y\}$ and consider $f \in \mathbf{S}_{Y^*}$ so that $f(x) = f(y) = 1$. Now, by hypothesis there must exist $g \in \mathbf{S}_{X^*}$ such that $g(x) = 1 > g(y)$. Next, $g|_Y \in \mathbf{S}_{Y^*}$, $g|_Y(x) = 1$, and $g|_Y \neq f$ so x is not a smooth point of $\mathbf{B}_Y$. This proves that $x \in \mathrm{nsmo}(\mathbf{B}_X)$. In the second place, assume that x is either a rotund point of $\mathbf{B}_X$ or a strongly non-smooth point of $\mathbf{B}_X$. If x is a rotund point of the unit ball of X, then x is a rotund point of the unit ball of every 2-dimensional subspace containing x. So, assume that $x \in \mathrm{nsmo}(\mathbf{B}_X)$. Let Y be a 2-dimensional subspace containing x. If x is a rotund point of $\mathbf{B}_Y$, then x is an exposed point of $\mathbf{B}_Y$. If x is not, then by hypothesis x is not a smooth point of $\mathbf{B}_Y$, therefore, according to Theorem 2.1, x is an exposed point of $\mathbf{B}_Y$. In the third and last place, assume that, for every 2-dimensional subspace Y containing x, x is an exposed point of $\mathbf{B}_Y$. Let $y \in \bigcap\{C \subseteq \mathbf{S}_X : C$ is an exposed face of $\mathbf{B}_X$ and $x \in C\}$ and suppose that $y \neq x$. Then, consider $Y = \mathrm{span}\{x, y\}$. Let $f \in \mathbf{S}_{Y^*}$ be such that $f(x) = f(y) = 1$. If $g \in \mathbf{S}_{Y^*}$ and $g(x) = 1$, then by hypothesis $g(y) = 1$ (to see this, consider a Hahn–Banach extension of g with norm 1). Since Y has dimension 2, $g = f$. This proves that x is a smooth point of $\mathbf{B}_Y$. However, x is not a rotund point of $\mathbf{B}_Y$ since $[x, y] \subset \mathbf{S}_Y$. As a consequence, we deduce that x is not an exposed point of $\mathbf{B}_Y$.

Lemma 2.4 Let X be a real Banach space. Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of exposed faces of $\mathbf{S}_X$. If $\bigcap_{n\in\mathbb{N}} C_n \neq \varnothing$, then $\bigcap_{n\in\mathbb{N}} C_n$ is an exposed face of $\mathbf{B}_X$. In particular, if X is separable, then the non-empty intersection of exposed faces is always an exposed face.

Proof For every $n \in \mathbb{N}$, there exists $f_n \in \mathbf{S}_{X^*}$ such that $C_n = f_n^{-1}(1) \cap \mathbf{B}_X$. Now, $f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n \in \mathbf{S}_{X^*}$ and $\bigcap_{n\in\mathbb{N}} C_n = f^{-1}(1) \cap \mathbf{B}_X$.

Theorem 2.5 Let X be a real Banach space. Then, $\exp(\mathbf{B}_X) \subseteq \operatorname{rot}(\mathbf{B}_X) \cup \operatorname{nsmo}_s(\mathbf{B}_X)$. If X is separable, then $\exp(\mathbf{B}_X) = \operatorname{rot}(\mathbf{B}_X) \cup \operatorname{nsmo}_s(\mathbf{B}_X)$.

Proof If $x \in \exp(\mathbf{B}_X)$, then x is an exposed point of the unit ball of every 2-dimensional subspace containing it, thus by Lemma 2.3, $x \in \operatorname{rot}(\mathbf{B}_X) \cup \operatorname{nsmo}_s(\mathbf{B}_X)$. Conversely, assume that X is separable. If $x \in \operatorname{rot}(\mathbf{B}_X) \cup \operatorname{nsmo}_s(\mathbf{B}_X)$, then, according to Lemma 2.3, $\{x\} = \bigcap\{C \subseteq \mathbf{S}_X : C$ is an exposed face of $\mathbf{B}_X$ and $x \in C\}$. Now, by applying Lemma 2.4, $\{x\}$ is an exposed face of $\mathbf{B}_X$.

Again unfortunately, there are strongly non-smooth points that are not exposed points, in other words, Theorem 2.5 does not hold in general.

Lemma 2.6 Let K be a compact Hausdorff topological space with more than one point. Then, the constant function f equal to 1 is a strongly non-smooth point of the unit ball of $\mathcal{C}(K)$.

Proof Let $g \in \mathcal{C}(K)$ with norm 1 such that $f \neq g$ and $\|\frac{f+g}{2}\|_\infty = 1$. In order to see that f is not a smooth point of the unit ball of $\operatorname{span}\{f, g\}$, it suffices to find a probability measure τ on K with $\int_K g \mathrm{d}\tau < 1$. There exists an open set U of K and $\varepsilon > 0$ such that $g(t) < 1 - \varepsilon$ for every $t \in U$. Now, let τ be a probability measure on K with $\tau(U) > 0$. We have that

$$\int_K g \mathrm{d}\tau = \int_U g \mathrm{d}\tau + \int_{K\setminus U} g \mathrm{d}\tau \leq (1 - \varepsilon)\tau(U) + \tau(K \setminus U) < 1.$$

Theorem 2.7 Let L be an uncountably infinite discrete topological space. Let us denote by $\widehat{L}$ the one-point compactification of L. Then, the constant function f equal to 1 is a strongly non-smooth point of the unit ball of $\mathcal{C}(\widehat{L})$ but not an exposed point.

Proof From Lemma 2.6 we already know that f is a strongly non-smooth point. Let us see that it is not an exposed point. Take any probability measure τ on $\widehat{L}$. Since L is uncountable, there must exist $t \in L$ with $\tau(\{t\}) = 0$. Now, it suffices to define $g : \widehat{L} \longrightarrow \mathbb{R}$ by $g(t) = 0$ and $g(\widehat{L} \setminus \{t\}) = \{1\}$. We have that $\|g\|_\infty = 1$ and $\int_{\widehat{L}} g \mathrm{d}\tau = 1$.

Finally, let us see that, by means of the following examples, we cannot consider the uniformly non-smooth points as a set of non-smooth points to characterize exposed points.

Definition 2.8 Let X be a real Banach space. A point $x \in \mathbf{S}_X$ will be said to be a uniformly non-smooth point of $\mathbf{B}_X$ if, for every $y \in \mathbf{S}_X \setminus \{x\}$, x is not a smooth point of $\mathbf{B}_Y$ where $Y = \operatorname{span}\{x, y\}$. The set of uniformly non-smooth points of $\mathbf{B}_X$ will be denoted by $\operatorname{nsmo}_u(\mathbf{B}_X)$.

Observe that $\operatorname{nsmo}_u(\mathbf{B}_X) \subseteq \operatorname{nsmo}_s(\mathbf{B}_X) \subseteq \operatorname{nsmo}(\mathbf{B}_X)$. Canonically, the uniformly non-smooth points compose the "smallest" possible set of non-smooth points. As we will show right away, this set is "too small".

Remark 2.9 (1) For any compact Hausdorff topological space K with more than one point, the constant function f equal to 1 is a uniformly non-smooth point of the unit ball of $\mathcal{C}(K)$. Indeed, let $g \in \mathcal{C}(K)$ with norm 1 so that $f \neq g$. We can assume, without loss of generality, that $\sup(g(K)) = 1$. Consider an open set U of K and $\varepsilon > 0$ so that $g(t) < 1-\varepsilon$ for every $t \in U$. Let τ be a probability measure on K so that $\tau(U) > 0$. We already know from the proof of Lemma 2.6 that $\int_K g\mathrm{d}\tau < 1$. Now, let $s \in K$ such that $g(s) > \int_K g\mathrm{d}\tau$, and consider another probability measure μ on K such that $\mu(\{s\}) = 1$. We have that $\int_K g\mathrm{d}\mu = g(s) > \int_K g\mathrm{d}\tau$. This proves that f is a uniformly non-smooth point of the unit ball of $\mathcal{C}(K)$. Therefore, if L denotes an uncountably infinite discrete topological space with one-point compactification $\widehat{L}$, then

$$\operatorname{rot}\left(\mathbf{B}_{\mathcal{C}(\widehat{L})}\right) \cup \operatorname{nsmo}_u\left(\mathbf{B}_{\mathcal{C}(\widehat{L})}\right) \nsubseteq \exp\left(\mathbf{B}_{\mathcal{C}(\widehat{L})}\right).$$

(2) Let us consider $\mathbb{R}^3$ with the norm whose unit ball is the intersection of the Euclidean unit ball with the set

$$\left\{(x,y,z) \in \mathbb{R}^3 : -\frac{1}{2} \leq z \leq \frac{1}{2}\right\}.$$

The point $\left(0, \sqrt{3}/2, 1/2\right)$ is an exposed point of this unit ball, but not a uniformly non-smooth point. Therefore, if we let X denote $\mathbb{R}^3$ endowed with the norm given by this unit ball, then

$$\exp(\mathbf{B}_X) \nsubseteq \operatorname{rot}(\mathbf{B}_X) \cup \operatorname{nsmo}_u(\mathbf{B}_X).$$

3 Conclusions

As a consequence of the previous section, the following questions are (negatively) answered:

(1) Is the non-empty intersection of exposed faces an exposed face? It is well known that the non-empty intersection of faces is always a face. Lemmas 2.3 and 2.6 and Theorem 2.7 show the existence of a non-exposed face which is the intersection of exposed faces.

(2) Is "being an exposed point" a 2-dimensional property? It is well known that both "being an extreme point" and "being a rotund point" are 2-dimensional properties. Lemmas 2.3 and 2.6 and Theorem 2.7 show the existence of a non-exposed point so that it is an exposed point of the unit ball of every 2-dimensional subspace containing it.

References

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