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On the relationship of location-independent riskier order to the usual stochastic order

Miguel A. Sordo *

Departamento de Estadística e Investigación Operativa, Facultad de Ciencias Económicas y Empresariales, Universidad de Cádiz, 11002 Cádiz, Spain

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ABSTRACT

Location-independent riskier order and its dual version, excess wealth order, compare random variables in terms of dispersion. In this note, we derive the relationship of both orders to the usual stochastic order. Some new properties of these orders are obtained as a consequence.

1. Introduction

Stochastic orders play an important role in reliability theory, risk theory and other fields which use as a tool the theory of probability. Over the years, a variety of stochastic orders have been proposed, giving rise to a large body of literature. Surveys can be found in Shaked and Shanthikumar (2007) and Müller and Stoyan (2002).

The most commonly used order to compare the magnitude of two random variables is the usual stochastic order. If F and G are the distribution functions of two random variables X and Y, respectively, the usual stochastic order is defined as follows.

Definition 1. Given two random variables X and Y, we say that X is smaller that Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $F(x) \geq G(x)$ for all real x.

On the other hand, dispersive orders describe when one random variable is more dispersed than another. Let F^{-1} and G^{-1} denote the quantile functions associated with F and G, respectively, defined by $F^{-1}(t) = \inf\{x : F(x) \geq t\}$ and $G^{-1}(t) = \inf\{x : G(x) \geq t\}$, for all $t \in (0, 1)$. The so-called dispersive order (see Doksum (1969) and Shaked (1982)) is defined as follows.

Definition 2. Given two random variables X and Y, we say that X is smaller that Y in the dispersive order (denoted by $X \leq_{\text{disp}} Y$) if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \quad \text{for all } 0 < \alpha \leq \beta < 1. \tag{1}$$

The relationship of dispersive order to usual stochastic order is well-known. Denote the supports of X and Y by supp(X) and supp(Y), respectively. If $l_X = \inf\{x : x \in \text{supp}(X)\}$, $u_X = \sup\{x : x \in \text{supp}(X)\}$ and l_Y and u_Y are similarly defined, we have the following result (see, for example, Theorem 3.B.13 in Shaked and Shanthikumar (2007)).

* Tel.: +34 956 015403; fax: +34 956 016288.
E-mail address: mangel.sordo@uca.es.

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Theorem 3. *Let X and Y be two random variables. Then,*

(a) *If $X \leq_{\text{disp}} Y$ and $-\infty < l_X \leq l_Y$, then $X \leq_{st} Y$.*
(b) *If $X \leq_{\text{disp}} Y$ and $u_Y \leq u_X < \infty$, then $X \geq_{st} Y$.*

Although dispersive order provides an important tool for comparing two random variables in terms of dispersion, (1) is a strong requirement and many pair of distributions can fail to satisfy it. This justifies the convenience of employing weaker orders to compare the dispersion of random variables. This paper deals with two of them, namely, the location-independent riskier order and its dual version, the excess wealth order, whose definitions are recalled here.

Definition 4. Let X and Y be two random variables. Then,

(i) X is said to be smaller than Y in the location-independent riskier order (denoted by $X \leq_{lir} Y$) if $E[v(Y)] = E[v(X-c)]$ implies $E[u(Y)] = E[u(X-c)]$ for all increasing concave functions u and v with u more risk averse than v (that is, $u = h(v)$, with h increasing and concave).
(ii) X is said to be smaller than Y in the excess wealth order (denoted by $X \leq_{ew} Y$) if $E\left[\left(X - F^{-1}(p)\right)^+\right] \leq E\left[\left(Y - G^{-1}(p)\right)^+\right]$ for all $p \in (0, 1)$, for which the expectations exist.

The location-independent riskier order was introduced by Jewitt (1989) to compare random assets in risk analysis. Landsberger and Meilijson (1994), who gave an interpretation of dispersive order in risk analysis, proved that $X \leq_{\text{disp}} Y$ implies $X \leq_{lir} Y$. For properties of the excess wealth order (also called the "right spread" order) we refer to Fernández-Ponce et al. (1998) and Shaked and Shanthikumar (1998). Some recent applications of this order in risk theory can be found in Sordo (2008). It is shown in Fagiuoli et al. (1999) that

$$X \leq_{lir} Y \iff -X \leq_{ew} -Y \tag{2}$$

and it is also well-known that $X \leq_{\text{disp}} Y$ implies $X \leq_{ew} Y$.

In this note we obtain the relationship of both location-independent riskier order and excess wealth order to the usual stochastic order. In Section 2 we provide a result that is stronger than Theorem 3; as a consequence, we strengthen some well-known properties of the dispersive order and give a new insight on the role played by the location-independent riskier order in reliability theory.

2. Some relationships among stochastic orders

We need the following result before obtaining the main results of this section.

Theorem 5. *Let X and Y be two random variables with distribution functions F and G, respectively. Then,*

(a) *$X \leq_{lir} Y$ if, and only if,*

$$F^{-1}(u) - G^{-1}(u) \leq \frac{1}{p}\int_0^p \left[F^{-1}(t) - G^{-1}(t)\right] \mathrm{d}t, \quad \forall 0 < p \leq u < 1. \tag{3}$$

(b) *$X \leq_{ew} Y$ if, and only if,*

$$F^{-1}(u) - G^{-1}(u) \geq \frac{1}{1-p}\int_p^1 \left[F^{-1}(t) - G^{-1}(t)\right] \mathrm{d}t, \quad \forall 0 < u \leq p < 1. \tag{4}$$

Proof. We give the proof of (a) (the proof of (b) follows from (a) by using (2)). Jewitt (1989) proved that $X \leq_{lir} Y$ if, and only if,

$$\frac{1}{u}\int_0^u \left[F^{-1}(t) - G^{-1}(t)\right] \mathrm{d}t \text{ is non-increasing in } u \in (0, 1). \tag{5}$$

By differentiation, it is seen that (5) is the same thing as requiring that

$$F^{-1}(u) - G^{-1}(u) \leq \frac{1}{u}\int_0^u \left[F^{-1}(t) - G^{-1}(t)\right] \mathrm{d}t, \quad \text{for all } u \in (0, 1). \tag{6}$$

If $X \leq_{lir} Y$, then (3) follows from (5) and (6). Conversely, by taking $u = p$ in (3), we obtain (6), which means $X \leq_{lir} Y$. ■

Now it is easy to derive the relationship of both location-independent riskier order and excess wealth order to the usual stochastic order. The following result is a stronger version of Theorem 3.

Theorem 6. *Let X and Y be two random variables. Then,*

(a) *If $X \leq_{lir} Y$ and $-\infty < l_X \leq l_Y$, then $X \leq_{st} Y$.*
(b) *If $X \leq_{ew} Y$ and $u_Y \leq u_X < \infty$, then $X \geq_{st} Y$.*

Proof. We first give the proof of (a). Letting $p \to 0^+$ in (3) we get

$$F^{-1}(u) - G^{-1}(u) \le l_X - l_Y. \tag{7}$$

From (7) and the assumptions on l_X and l_Y it follows $F^{-1}(u) \le G^{-1}(u)$ for all u, and this means $X \le_{st} Y$. The proof of (b) is similar by letting $p \to 1^-$ in (4). ■

Li and Shaked (2007) have also established that location-independent riskier order implies the usual stochastic order for the case of continuous random variables having 0 as the left endpoint of their supports. Note that Theorem 6(a) is more general, since we do not impose constraints on the class of random variables to be compared, except that $-\infty < l_X \le l_Y$.

It is well-known (see Theorem 3.B.14 in Shaked and Shanthikumar (2007)), that two random variables having the same finite interval as their supports are not related in terms of the dispersive order unless they are identical. A stronger result is the following.

Corollary 7. *Let X and Y be two random variables having the same finite support S. If $X \le_{lir} Y$ or $X \le_{ew} Y$, then X and Y have the same distribution.*

Proof. First, we suppose that $X \le_{lir} Y$. Since $l_X = l_Y > -\infty$, it follows from Theorem 6 that $X \le_{st} Y$. On the other hand, letting $u \to 1^-$ in (3) and taking into account that $u_X = u_Y < \infty$, we get

$$\int_0^p \left[F^{-1}(t) - G^{-1}(t)\right] dt \ge 0, \quad \text{for all } p \in (0, 1),$$

which means $X \ge_{icv} Y$, where $\ge_{icv}$ denotes the increasing concave order (see, for example, Sordo and Ramos (2007)). Now it is easy to see that $X \le_{st} Y$ and $X \ge_{icv} Y$ will hold if, and only if, X and Y have the same distribution. The proof is similar under the assumption $X \le_{ew} Y$. ■

Using the same kind of argument it is easy to prove the following result, which is stronger than Theorem 3.B.15 in Shaked and Shanthikumar (2007).

Corollary 8. *Let X and Y be two random variables whose supports are intervals. If $X \le_{lir} Y$ or $X \le_{ew} Y$, then $\mu\{\text{supp}(X)\} \le \mu\{\text{supp}(Y)\}$, where μ denotes the Lebesgue measure.*

Dispersive order and the usual stochastic order are specially relevant to life distributions. The reason is that these orders characterize, by comparisons of residual lives at different times, the IFR (increasing failure rate) and DFR (decreasing failure rate) aging notions. In order to provide an application of Theorem 6(a), let X be the lifetime of an item and let X_t be the residual life of X at t, given by $X_t \equiv [X - t \mid X > t]$. The next result follows from the mentioned characterizations by using Theorem 6(a) and the relationship of location-independent riskier order to dispersive order.

Theorem 9. *Let X be a random variable with continuous distribution function F and support $S = (k, \infty)$, where $k \ge -\infty$ [respectively, $k > -\infty$]. Then, X is IFR [DFR] if, and only if, $X_s \ge_{lir} [\le_{lir}] X_t$, for all $s < t$.*

Theorem 9 shows that the role played by the location-independent riskier order in reliability theory is, despite (2), quite different from the role played by the excess wealth order, which, as shown by Belzunce (1999), characterizes the DMRL and IMRL aging notions.

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