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Convex ordered median problem with ℓ_p-norms

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ABSTRACT

This paper presents a procedure to solve the convex ordered median problem where the distances are measured with ℓ_p-norms. In order to do that, we consider an approximated problem and develop an algorithm based on a gradient descent method that generates a sequence with decreasing objective value. We prove its convergence to the optimal solution of the approximated problem. The paper ends with some computational results of the proposed methodology.

1. Introduction

Location analysis is one of the most active fields in Operations Research; in fact, many different models have been developed in the last decades to deal with different real world situations. Notice that a very important aspect of a location model is the correct choice of the objective function and in most classical location models the objective function is the main differentiator. The median objective is to minimize the sum of the weighted distances from the clients to the server. The center objective is to minimize the maximum weighted distance from a client to the server. The cent-dian objective is a convex combination of the median and center objectives; it aims to keep both the average cost behavior as well as the highest cost in balance. Despite the fact that all three objectives (and some more) are frequently encountered in the literature (see for example [1]), not much has been done in the direction of a unified framework for handling all of these objectives.

The increasing need for location models to better fit different real situations, has made it necessary to develop new and flexible location models. To that end, [2] introduced a new type of objective function that generalizes the most popular objective functions mentioned above. This objective function, called ordered median function, applies a penalty to the weighted distance from a client to the server, which is dependent on the position of that weighted distance in the vector of all weighted distances from the clients to the server. For example, a different penalty might be applied to a client if the weighted distance to the server is in the 5th-position rather than in the 2nd-position. It is even possible to neglect some customers by assigning a zero penalty. This adds a "sorting"-problem to the underlying facility location problem, making formulation and solution much more challenging.

In the last years, these flexible objective functions have attracted the attention of researchers. Puerto and Fernández [2,3] studied characterizations of the solution set for the general formulations. For the planar case with polyhedral gauges, [4] develops a polynomial time algorithm and [5] applies these models to semiobnoxious location problems. In network location problems, efforts have been devoted to obtaining finite dominating sets and efficient algorithms to solve this kind of problems [6–10]. Recently, the discrete versions of these models have also been studied in [11–14]. Research in this area also led to a recent monograph; see [15].

However, in continuous location theory, these models have only dealt with smooth norms in [16] for the Euclidean case. In this paper, we will consider these formulations when the distances are measured with ℓ_p-norms. Notice that the measurement of the distances with ℓ_p-norms better fits to some real world situations (see [17,18]). In particular, we will restrict ourselves to convex ordered median problems and $p \in [1,2]$, similarly to other published studies for the median problem, e.g. [19–24]. (Although we can find some papers in the literature dealing with the median problem for $p > 2$, the solution procedure developed in those papers does not guarantee a global convergence to an optimal solution; see [25,26].) For this type of problems, we will develop an iterative procedure based on a modified gradient descent method. Observe that this methodology is complex because the objective function does not have a common expression as sum of the weighted distances from the clients to the server; in fact, it is pointwise defined. On the other hand, this procedure is very robust because we provide a common tool to solve different classical models, for instance, median problems, center problems, cent-dian problems, among others and new ones that can be modelled under this formulation. Moreover, we are also providing a method to solve well-known models for which currently no resolution method has been published, such as the k-centrum problem.

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The paper is organized as follows. In the next section, we present the convex ordered median problem when the distances are measured with ℓ_p-norms. In Section 3, we introduce the approximated ordered median problem through the hyperbolic approximation. In Section 4, we develop a procedure for solving this problem. The proposed algorithm and the convergence to the optimal solution are given in Section 5. The paper ends with some computational results and conclusions.

2. The model

In this section we present the convex ordered median problem where the distances are measured with an ℓ_p-norm. Consider the set of demand points in the plane $A=\{a_1,\dots,a_M\}$ and two sets of non-negative scalars $\Omega:=\{\omega_1,\dots,\omega_M\}$ and $\Lambda:=\{\lambda_1,\dots,\lambda_M\}$, where $\lambda_1\leqslant\cdots\leqslant\lambda_M$ (as we will see later, this condition on the λ-weights provides the convexity property of the ordered median function). The elements ω_i are weights corresponding to the importance given to the existing facilities $a_i,\ i\in\{1,\dots,M\}$ and the elements of Λ allow one to choose among different kinds of objective functions. Given a permutation σ of $\{1,\dots,M\}$ verifying

$$\omega_{\sigma_1}\|x-a_{\sigma_1}\|_p\leqslant\cdots\leqslant\omega_{\sigma_M}\|x-a_{\sigma_M}\|_p,$$

where $\|\cdot\|_p$ denotes the ℓ_p-norm, we define

$$d_{(i)}(x):=\omega_{\sigma_i}\|x-a_{\sigma_i}\|_p.$$

Notice that the order of the sequence depends on the point x.

The convex ordered median problem is given by the following formulation:

$$\min_{x\in\mathbb{R}^2} F(x)=\sum_{i=1}^{M}\lambda_i d_{(i)}(x). \tag{1}$$

For different choices of λ we obtain different types of objective functions that include the classical location problems as median problems ($\lambda=(1,1,\dots,1)$), center problems ($\lambda=(0,0,\dots,0,1)$), μ-cent-dian problems ($\lambda=(\mu,\mu,\dots,\mu,1)$ for $0<\mu<1$) and k centrum problems ($\lambda=(0,\dots,0,\overbrace{1,\dots,1}^{k})$). Moreover, as already mentioned in the Introduction, new useful objective functions can easily be modelled under this formulation. For example, when locating a distribution center of perishable goods where the goal is for the longer distances and the total travel distance to be as small as possible, we might consider the following λ vector, $\lambda=(1,2,\dots,M)$.

We can also see that this objective function is pointwise defined. This means that the objective function has different explicit expressions as sum of the weighted distances to the demand points, depending on the order of the sequence of the weighted distances.

Example 2.1. Consider two demand points $a_1=(4,1)$ and $a_2=(1,0)$. For $p=1.5,\ \omega_1=\omega_2=1$ and $\lambda_1=1,\ \lambda_2=2$, we have that $F(x)$ has two different expressions as sum of the weighted distances at the points $x^q=(2.8,0.4)$ and $x^{q+1}=(2.24,0.32)$ (see Fig. 1).

Indeed, since $\omega_1\|x^q-a_1\|_p=1.47<1.92=\omega_2\|x^q-a_2\|_p$ we get

$$\begin{aligned}F(x^q)&=\lambda_1\cdot\omega_1\|x-a_1\|_p+\lambda_2\cdot\omega_2\|x-a_2\|_p\\&=\|(2.8,0.4)-(4,1)\|_{1.5}+2\cdot\|(2.8,0.4)-(1,0)\|_{1.5}\\&=5.32.\end{aligned}$$

Moreover, since $\omega_1\|x^{q+1}-a_1\|_p=2.036>1.341=\omega_2\|x^{q+1}-a_2\|_p$ we get

$$\begin{aligned}F(x^{q+1})&=\lambda_2\cdot\omega_1\|x^{q+1}-a_1\|_p+\lambda_1\cdot\omega_2\|x^{q+1}-a_2\|_p\\&=2\cdot\|(2.24,0.32)-(4,1)\|_{1.5}+\|(2.24,0.32)-(1,0)\|_{1.5}\\&=5.41.\end{aligned}$$

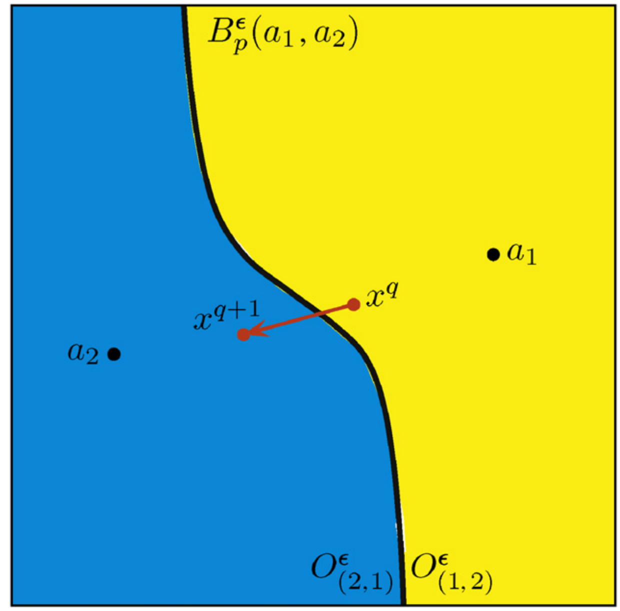

Fig. 1. Bisector of a_1 and a_2.

In addition, since the λ-weights are given in non-decreasing order, we obtain a reformulation for $F(x)$ (see [4]):

$$F(x)=\max_{\sigma\in\mathscr{P}(M)} F_\sigma(x),$$

where $\mathscr{P}(M)$ stands for the set of permutations of $\{1,\dots,M\}$ and $F_\sigma(x)=\sum_{i=1}^{M}\lambda_i\omega_{\sigma_i}\|x-a_{\sigma_i}\|_p$. Hence, since $F_\sigma(x)$ is a convex function for each $\sigma\in\mathscr{P}(M)$, $F(x)$ is also convex (observe that $\lambda_1\leqslant\cdots\leqslant\lambda_M$ is also a necessary condition for the convexity of $F_\sigma(x)$, see [15,27]).

In order to obtain a better understanding of the ordered median problem, we give some definitions. (For further details the reader is referred to [15].)

Definition 2.1. Given $a_i,a_j\in A$, the bisector of a_i and a_j with respect to $\|\cdot\|_p$ is defined as the set $B_p(a_i,a_j)=\{x\in\mathbb{R}^2:\ \omega_i\|x-a_i\|_p=\omega_j\|x-a_j\|_p\}$.

Definition 2.2. Given $\sigma\in\mathscr{P}(M)$, the ordered region O_σ is defined by

$$O_\sigma=\{x\in\mathbb{R}^2:\ \omega_{\sigma_1}\|x-a_{\sigma_1}\|_p\leqslant\cdots\leqslant\omega_{\sigma_M}\|x-a_{\sigma_M}\|_p\}.$$

As an illustration of Definitions 2.1 and 2.2 we can see the bisector line for the points a_1 and a_2 with $p=1.5$ in Fig. 1 (note that $B_p(a_i,a_j)=B_p(a_j,a_i)$). Moreover, this bisector divides the plane into two ordered regions $O_{(1,2)}$ and $O_{(2,1)}$. Observe that the ordered regions are not necessarily convex.

The importance of the ordered regions is that the ordered median objective function behaves like a classical median objective function in these regions ($F(x)=F_\sigma(x)$ for all $x\in O_\sigma$ and $\sigma\in\mathscr{P}(M)$).

In what follows, we denote the interior of a set S by $\mathrm{int}(S)$.

3. The approximated ordered median problem

The solution procedure that we propose in this paper is based on a modified version of the gradient descent method. However, since the objective function of Problem (1) is not differentiable at the demand points (nor in the bisector lines), we will use the so-called hyperbolic approximation (see [20,21,23,28]).

Consider $h_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ given by $h_\varepsilon(v)^{\mathrm{T}} := (h_{\varepsilon,1}(v), h_{\varepsilon,2}(v))$ and $h_{\varepsilon,j}(v) := (v_j^2+\varepsilon^2)^{1/2}$, for $j=1,2$ and $\varepsilon>0$ fixed. Problem (1) is then approximated by

$$\min_{x\in\mathbb{R}^2} F_\varepsilon(x) := \sum_{i=1}^{M} \lambda_i d^\varepsilon_{(i)}(x), \tag{2}$$

where $d^\varepsilon_{(i)}(x)=\omega_{\beta_i}\|h_\varepsilon(x-a_{\beta_i})\|_p$ (weighted approximated distances) for some $\beta\in\mathscr{P}(M)$ such that

$$\omega_{\beta_1}\|h_\varepsilon(x-a_{\beta_1})\|_p \leqslant \cdots \leqslant \omega_{\beta_M}\|h_\varepsilon(x-a_{\beta_M})\|_p.$$

Since $\|h_\varepsilon(x-a_{\beta_i})\|_p$ is strictly convex (see [23]), and $\lambda_1\leqslant\cdots\leqslant\lambda_M$, we have that $F_\varepsilon(x)$ is the maximum of strictly convex functions, and consequently it is also strictly convex. Thus, Problem (2) has a unique optimal solution.

We define the approximated bisector of a_i and a_j as

$$B^\varepsilon_p(a_i,a_j)=\{x\in\mathbb{R}^2 : \omega_i\|h_\varepsilon(x-a_i)\|_p=\omega_j\|h_\varepsilon(x-a_j)\|_p\}.$$

For a given $\beta\in\mathscr{P}(M)$, the approximated ordered region is

$$O^\varepsilon_\beta=\{x\in\mathbb{R}^2 : \omega_{\beta_1}\|h_\varepsilon(x-a_{\beta_1})\|_p\leqslant\cdots\leqslant\omega_{\beta_M}\|h_\varepsilon(x-a_{\beta_M})\|_p\}$$

and $F_{\varepsilon,\beta}(x) := \sum_{i=1}^M \lambda_i\omega_{\beta_i}\|h_\varepsilon(x-a_{\beta_i})\|_p$. Observe that $F_\varepsilon(x)=F_{\varepsilon,\beta}(x)$ for any $x\in O^\varepsilon_\beta$ with $\beta\in\mathscr{P}(M)$. The error caused by replacing the ordered median problem with the approximated problem can be bounded as the following result shows.

Lemma 3.1. *The following property is satisfied for any* $x\in\mathbb{R}^2$

$$|F_\varepsilon(x)-F(x)|\leqslant\left(\sum_{i=1}^M w_i\right)\lambda_M\, 2^{1/p}\,\varepsilon^{1/2}.$$

Proof. Given $\sigma,\beta\in\mathscr{P}(M)$, let $\sigma^{-1},\beta^{-1}\in\mathscr{P}(M)$ be such that

$$F(x)=\sum_{i=1}^M\lambda_i\omega_{\sigma_i}\|x-a_{\sigma_i}\|_p=\sum_{i=1}^M\lambda_{\sigma^{-1}_{(i)}}\omega_i\|x-a_i\|_p,$$
$$F_\varepsilon(x)=\sum_{i=1}^M\lambda_i\omega_{\beta_i}\|h_\varepsilon(x-a_{\beta_i})\|_p=\sum_{i=1}^M\lambda_{\beta^{-1}_{(i)}}\omega_i\|h_\varepsilon(x-a_i)\|_p.$$

Moreover, we have that (see [22] for further details)

$$\|h_\varepsilon(x-a_i)\|_p-\|x-a_i\|_p\leqslant 2^{1/p}\varepsilon^{1/2}.$$

Therefore, since $\lambda_1\leqslant\cdots\leqslant\lambda_M$, we get

$$\begin{aligned}|F_\varepsilon(x)-F(x)|&\leqslant\sum_{i=1}^M|\lambda_{\beta^{-1}_{(i)}}\omega_i\|h_\varepsilon(x-a_i)\|_p-\lambda_{\sigma^{-1}_{(i)}}\omega_i\|x-a_i\|_p|\\&\leqslant\left(\sum_{i=1}^M w_i\right)\lambda_M\,2^{1/p}\,\varepsilon^{1/2}.\qquad\square\end{aligned}$$

We now analyze the case that x^* is the optimal solution for Problem (2) and $x^*\in\mathrm{int}(O^\varepsilon_\beta)$ for some $\beta\in\mathscr{P}(M)$. (Observe that the optimal solution of Problem (2) does not necessarily belong to the interior of an approximated ordered region.) Since $F_\varepsilon(x)$ is differentiable for $x\in\mathrm{int}(O^\varepsilon_\beta)$, x^* must satisfy $\nabla F_\varepsilon(x^*)=0$. The jth component, $\nabla_j F_\varepsilon(x)$, of the gradient vector $\nabla F_\varepsilon(x)$ for $x\in\mathrm{int}(O^\varepsilon_\beta)$ is

$$\begin{aligned}\nabla_j F_\varepsilon(x)=&\sum_{i=1}^M\lambda_i\cdot\omega_{\beta_i}\cdot\|h_\varepsilon(x-a_{\beta_i})\|_p^{1-p}\cdot h^{p-2}_{\varepsilon j}(x-a_{\beta_i})\\&\times(x_j-a_{\beta_i j}),\quad j=1,2.\end{aligned}$$

Setting these partial derivatives equal to zero and isolating the n components of the optimal solution x^* yields

$$x_j^*=\frac{\sum_{i=1}^M\lambda_i\omega_{\beta_i}\cdot\|h_\varepsilon(x^*-a_{\beta_i})\|_p^{1-p}\cdot h^{p-2}_{\varepsilon j}(x^*-a_{\beta_i})\cdot a_{\beta_i j}}{\sum_{i=1}^M\lambda_i\omega_{\beta_i}\cdot\|h_\varepsilon(x^*-a_{\beta_i})\|_p^{1-p}\cdot h^{p-2}_{\varepsilon j}(x^*-a_{\beta_i})}$$
$$j=1,2.$$

The equations above suggest the following point iterative scheme:

$$x_j^{q+1}=\frac{\sum_{i=1}^M\lambda_i\omega_{\beta_i}\cdot\|h_\varepsilon(x^q-a_{\beta_i})\|_p^{1-p}\cdot h^{p-2}_{\varepsilon j}(x^q-a_{\beta_i})\cdot a_{\beta_i j}}{\sum_{i=1}^M\lambda_i\omega_{\beta_i}\cdot\|h_\varepsilon(x^q-a_{\beta_i})\|_p^{1-p}\cdot h^{p-2}_{\varepsilon j}(x^q-a_{\beta_i})}$$
$$j=1,2, \tag{3}$$

where the superscript $q=0,1,2,\ldots,$ indicates the iteration number.

Notice that x^{q+1} is well defined for any $x^q\in\mathbb{R}^2$, even in the case that $x^*\notin\mathrm{int}(O^\varepsilon_\beta)$ for some $\beta\in\mathscr{P}(M)$ (obviously, in this case $\nabla F_\varepsilon(x^*)$ is not defined). Our goal in this paper is to develop a solution procedure for Problem (2) using the expression above.

Remark 3.1. This iterative scheme has an expression similar to the hyperbolic approximation Weiszfeld algorithm for the median problem (see [20]). However, the assignment of the λ-weights to the demand points is not fixed but depends on the ordered region to which x^q belongs. Indeed, this procedure does not guarantee that $x^{q+1}\in O^\varepsilon_\beta$ whenever $x^q\in\mathrm{int}(O^\varepsilon_\beta)$ (see Example 4.1).

On the other hand, for $x^q\in\mathrm{int}\,(O^\varepsilon_\beta)$, expression (3) can be rewritten as a modified gradient method, since

$$x_j^{q+1}=x_j^q-D_j^\beta(x^q),\quad j=1,2, \tag{4}$$

where

$$D_j^\beta(x^q)=C^{-1}_{\beta j}(x^q)\cdot\nabla_j F_{\varepsilon,\beta}(x^q),$$

$$\begin{aligned}\nabla_j F_{\varepsilon,\beta}(x^q)=&\sum_{i=1}^M\lambda_i\cdot\omega_{\beta_i}\cdot\|h_\varepsilon(x^q-a_{\beta_i})\|_p^{1-p}\cdot h^{p-2}_{\varepsilon j}(x^q-a_{\beta_i})\\&\times(x_j^q-a_{\beta_i j}),\end{aligned}$$

$$C_{\beta j}(x^q)=\sum_{i=1}^M\lambda_i\cdot\omega_{\beta_i}\cdot\|h_\varepsilon(x^q-a_{\beta_i})\|_p^{1-p}\cdot h^{p-2}_{\varepsilon j}(x^q-a_{\beta_i}). \tag{5}$$

Observe that $C_{\beta j}(x^q)>0$ for $j=1,2$. Moreover, the index β in the expressions above can be omitted, since in the case $x^q\in\mathrm{int}(O^\varepsilon_\beta)$ it is implicitly given by x^q. However, the index β has been kept to be consistent with the notation in the next sections for those cases where x^q does not belong to $\mathrm{int}(O^\varepsilon_\beta)$ for any $\beta\in\mathscr{P}(M)$.

4. Iterative procedure

In this section, we present a new iterative procedure to solve Problem (2). In order to do that, we will generate a sequence $\{x^q\}_{q\in\mathbb{N}}$ such that $F_\varepsilon(x^{q+1})<F_\varepsilon(x^q)$, for $q\geqslant 1$. Since for $x^q\in\mathrm{int}(O^\varepsilon_\beta)$ the iterative scheme given by (4) may generate x^{q+1} such that $x^{q+1}\notin O^\varepsilon_\beta$, the descent property ensured by the Weiszfeld algorithm for the median problem might not be satisfied.

Example 4.1. Consider Example 2.1, i.e., $a_1=(4,1)$ and $a_2=(1,0)$ are the demand points, $p=1.5$, $\omega_1=\omega_2=1$ and $\lambda_1=1$, $\lambda_2=2$.

For $x^q=(2.8,0.4)$ and $\varepsilon=0.001$, we have that $\omega_1\|h_\varepsilon(x^q-a_1)\|_p=1.47<1.92=\omega_2\|h_\varepsilon(x^q-a_2)\|_p$, then, $x^q\in \mathrm{int}(O^\varepsilon_{(1,2)})$.

Hence, using (4), we get that $x^{q+1}=(2.24,0.32)$. We can check that $\omega_1\|h_\varepsilon(x^{q+1}-a_1)\|_p=2.036>1.341=\omega_2\|h_\varepsilon(x^{q+1}-a_2)\|_p$, or equivalently, $x^{q+1}\in \mathrm{int}(O^\varepsilon_{(2,1)})$. Thus, this is an example where $x^q\in \mathrm{int}(O^\varepsilon_{(1,2)})$ and $x^{q+1}\notin O^\varepsilon_{(1,2)}$ (see Fig. 1).

Moreover, $F_\varepsilon(x^q)=5.32<5.41=F_\varepsilon(x^{q+1})$. Therefore, the iterative scheme given by (4) does not guarantee the descent property.

In the following, our goal will be to propose a procedure that allows us to obtain a sequence of points with decreasing objective value. In order to do that, we distinguish between the cases x^q in the interior and x^q on the boundary of an approximated ordered region.

4.1. x^q in the interior of an approximated ordered region

Let x^q be in the interior of an approximated ordered region O^ε_β with $\beta\in\mathscr{P}(M)$, that is,

$$\omega_{\beta_1}\|h_\varepsilon(x^q-a_{\beta_1})\|_p<\cdots<\omega_{\beta_M}\|h_\varepsilon(x^q-a_{\beta_M})\|_p.$$

In order to develop an iterative scheme that generates a sequence with decreasing objective value, we propose a modification of expression (4) by introducing the stepsize $K(x^q)$ as follows:

$$x_j^{q+1}=x_j^q-K(x^q)\cdot D_j^\beta(x^q),\quad j=1,2.$$

The following technical results allow us to define $K(x^q)$ such that $x^{q+1}\in O^\varepsilon_\beta$ for $x^q\in\mathrm{int}(O^\varepsilon_\beta)$.

Let $d_p(x,S)$ be the infimal distance function from a point x to the set S measured with a ℓ_p-norm, that is,

$$d_p(x,S)=\inf\{\|x-y\|_p : y\in S\}.$$

Lemma 4.1. *For any $a_i,a_j\in A$, $y\in B^\varepsilon_p(a_i,a_j)$ and $x\in\mathbb{R}^2$ we have that*

$$\|h_\varepsilon(x-y)\|_p\geqslant\frac{|\omega_i\|h_\varepsilon(x-a_i)\|_p-\omega_j\|h_\varepsilon(x-a_j)\|_p|}{\omega_i+\omega_j}.$$

Proof. Applying the triangular inequality we obtain that

$$\omega_i\|h_\varepsilon(x-y)\|_p\geqslant\omega_i\|h_\varepsilon(x-a_i)\|_p-\omega_i\|h_\varepsilon(y-a_i)\|_p,\quad \forall x,y,a_i\in\mathbb{R}^2.$$

Moreover, since $y\in B^\varepsilon_p(a_i,a_j)$, then $\omega_i\|h_\varepsilon(y-a_i)\|_p=\omega_j\|h_\varepsilon(y-a_j)\|_p$. Thus,

$$\omega_i\|h_\varepsilon(x-y)\|_p\geqslant\omega_i\|h_\varepsilon(x-a_i)\|_p-\omega_j\|h_\varepsilon(y-a_j)\|_p.$$

Again, applying the triangular inequality we have that

$$\omega_i\|h_\varepsilon(x-y)\|_p\geqslant\omega_i\|h_\varepsilon(x-a_i)\|_p-\omega_j\|h_\varepsilon(y-x)\|_p-\omega_j\|h_\varepsilon(x-a_j)\|_p,$$

or equivalently

$$\|h_\varepsilon(x-y)\|_p\geqslant\frac{\omega_i\|h_\varepsilon(x-a_i)\|_p-\omega_j\|h_\varepsilon(x-a_j)\|_p}{\omega_i+\omega_j},\quad \forall x\in\mathbb{R}^2,\ y\in B^\varepsilon_p(a_i,a_j).$$

Analogously, interchanging a_i and a_j we have that

$$\|h_\varepsilon(x-y)\|_p\geqslant\frac{\omega_j\|h_\varepsilon(x-a_j)\|_p-\omega_i\|h_\varepsilon(x-a_i)\|_p}{\omega_i+\omega_j},\quad \forall x\in\mathbb{R}^2,\ y\in B^\varepsilon_p(a_i,a_j)$$

and the result follows. □

Corollary 4.1. *For any $a_i,a_j\in A$, $B^\varepsilon_p(a_i,a_j)$ satisfies that*

$$d_p(x,B^\varepsilon_p(a_i,a_j))\geqslant\frac{|\omega_i\|h_\varepsilon(x-a_i)\|_p-\omega_j\|h_\varepsilon(x-a_j)\|_p|}{\omega_i+\omega_j}-2^{1/p}\varepsilon^{1/2},\quad\forall x\in\mathbb{R}^2.$$

Proof. First, by [22] we have that $\|h_\varepsilon(x-y)\|_p-\|x-y\|_p\leqslant 2^{1/p}\varepsilon^{1/2}$. Therefore, applying Lemma 4.1, we obtain that

$$\|x-y\|_p\geqslant\frac{|\omega_i\|h_\varepsilon(x-a_i)\|_p-\omega_j\|h_\varepsilon(x-a_j)\|_p|}{\omega_i+\omega_j}-2^{1/p}\varepsilon^{1/2},\quad \forall y\in B^\varepsilon_p(a_i,a_j),\ \forall x\in\mathbb{R}^2,$$

and the result follows. □

Notice that this lower bound makes sense if it is positive. Now, we will analyze the case where $x\in\mathrm{int}(O^\varepsilon_\beta)$, for some $\beta\in\mathscr{P}(M)$, and

$$\frac{\omega_{\beta_{t+1}}\|h_\varepsilon(x-a_{\beta_{t+1}})\|_p-\omega_{\beta_t}\|h_\varepsilon(x-a_{\beta_t})\|_p}{\omega_{\beta_{t+1}}+\omega_{\beta_t}}-2^{1/p}\varepsilon^{1/2}>0,\quad \forall t\in\{1,\dots,M-1\}. \tag{6}$$

After that, we will study the case where $t\in\{1,\dots,M-1\}$ exists such that condition (6) does not hold.

4.1.1. Condition (6) is fulfilled

Corollary 4.1 allows us to obtain a lower bound of the distance from any point $x\in\mathbb{R}^2$ to any approximated bisector line defined by two demand points. We will use this lower bound to obtain an iterative scheme providing new iterates in the same ordered region as the previous ones.

Given $x\in\mathrm{int}(O^\varepsilon_\beta)$, for some $\beta\in\mathscr{P}(M)$ and satisfying (6), we consider the following algorithmic map $\Phi_\beta(x)=(\Phi_{\beta,1}(x),\Phi_{\beta,2}(x))$, with iterates given by $x^{q+1}=\Phi_\beta(x^q)$, where

$$\Phi_{\beta,j}(x^q)=x_j^q-K(x^q)\cdot D_j^\beta(x^q),\quad j=1,2 \tag{7}$$

with $D^\beta(x^q)=(D_1^\beta(x^q),D_2^\beta(x^q))$ defined by (5) and

$$K(x^q)=\min\left\{1,\frac{1}{\|D^\beta(x^q)\|_p}\times\min_{t\in\{1,\dots,M-1\}}\left\{\frac{\omega_{\beta_{t+1}}\|h_\varepsilon(x^q-a_{\beta_{t+1}})\|_p-\omega_{\beta_t}\|h_\varepsilon(x^q-a_{\beta_t})\|_p}{\omega_{\beta_{t+1}}+\omega_{\beta_t}}-2^{1/p}\varepsilon^{1/2}\right\}\right\}.$$

The expression that defines $K(x^q)$ is justified by the following lemmas.

Lemma 4.2. *If $x^q\in\mathrm{int}(O^\varepsilon_\beta)$ verifies* (6), *then $x^{q+1}\in O^\varepsilon_\beta$.*

Proof. By definition of x^{q+1} and Corollary 4.1, we have the following inequalities:

$$\|x^{q+1}-x^q\|_p\leqslant\frac{|\omega_{\beta_{t+1}}\|h_\varepsilon(x^q-a_{\beta_{t+1}})\|_p-\omega_{\beta_t}\|h_\varepsilon(x^q-a_{\beta_t})\|_p|}{\omega_{\beta_{t+1}}+\omega_{\beta_t}}-2^{1/p}\varepsilon^{1/2}\leqslant d_p(x^q,B^\varepsilon_p(a_{\beta_t},a_{\beta_{t+1}})),$$

for any $t\in\{1,\dots,M-1\}$. Therefore, $x^{q+1}\in O^\varepsilon_\beta$. □

Lemma 4.3. *If $x^q\in\mathrm{int}(O^\varepsilon_\beta)$ verifies* (6) *and $\nabla F_{\varepsilon,\beta}(x^q)\neq 0$, then $F_\varepsilon(x^{q+1})<F_\varepsilon(x^q)$, $\forall q\geqslant 1$.*

Proof. By Lemma 4.2, we have that $x^q, x^{q+1} \in O^{\varepsilon}_{\beta}$, then $F_{\varepsilon}(x^q) = F_{\varepsilon,\beta}(x^q)$ and $F_{\varepsilon}(x^{q+1}) = F_{\varepsilon,\beta}(x^{q+1})$.

Depending on the value of $K(x^q)$ we distinguish two cases. If $K(x^q)=1$, then x^{q+1} is given by the classical iteration of the Weiszfeld algorithm for the hyperbolic approximation function $F_{\varepsilon,\beta}(\cdot)$ and in [20] it is proven that $F_{\varepsilon,\beta}(x^{q+1}) < F_{\varepsilon,\beta}(x^q)$.

If $K(x^q) \neq 1$, we have that x^{q+1} is obtained by a modified gradient method with stepsize lower than the Weiszfeld algorithm for the function $F_{\varepsilon,\beta}(\cdot)$. Hence, since the Weiszfeld algorithm gives a descent sequence, by convexity arguments we have that $F_{\varepsilon,\beta}(x^{q+1}) < F_{\varepsilon,\beta}(x^q)$. $\square$

Lemma 4.4. *Let $x^q \in \mathrm{int}(O^{\varepsilon}_{\beta})$ and denote x^* the optimal solution of Problem* (2).

(i) *If $\Phi_{\beta}(x^q) = x^q$ then $x^q = x^*$.*
(ii) *If $x^q = x^*$ then $\Phi_{\beta}(x^q) = x^*$.*

Proof. If $\Phi_{\beta}(x^q) = x^q$, by the definition of $\Phi_{\beta}(x^q)$ we have that $\nabla F_{\varepsilon,\beta}(x^q) = \nabla F_{\varepsilon}(x^q) = 0$ and then $x^q = x^*$. Conversely, if $x^q = x^*$, since $F_{\varepsilon}(x^q)$ is differentiable in the interior of an ordered region we have that $\nabla F_{\varepsilon}(x^q) = 0$. Therefore, by the definition of $\Phi_{\beta}(x^q)$ we obtain that $\Phi_{\beta}(x^q) = x^q$ and the result follows. $\square$

4.1.2. *Condition (6) is not fulfilled*

First, we define

$$T(x^q) = \{t \in \{1,\dots,M-1\} : x^q \text{ does not satisfy } (6)\}, \tag{8}$$

and let

$$\hat{\Phi}_{\beta j}(x^q) = x^q_j - \hat{K}(x^q) D^{\beta}_j(x^q), \quad j = 1,2, \tag{9}$$

where $D^{\beta}_j(x^q)$ is given by (5) and

$$\hat{K}(x^q) = \min\left\{1, \frac{1}{\|D^{\beta}(x^q)\|_p} \times \min_{t \notin T(x^q)} \left\{ \frac{\omega_{\beta_{t+1}} \|h_{\varepsilon}(x^q - a_{\beta_{t+1}})\|_p - \omega_{\beta_t} \|h_{\varepsilon}(x^q - a_{\beta_t})\|_p}{\omega_{\beta_{t+1}} + \omega_{\beta_t}} - 2^{1/p}\varepsilon^{1/2} \right\}\right\}.$$

If $\hat{\Phi}_{\beta}(x^q) \in O^{\varepsilon}_{\beta}$, then we define $x^{q+1} = \hat{\Phi}_{\beta}(x^q)$. Otherwise, we define $x^{q+1} = x^q_{\beta}$ as the intersection point closest to x^q of the segment $[x^q, \hat{\Phi}_{\beta}(x^q)]$ with some bisector $B^{\varepsilon}_p(a_t, a_{t+1})$ with $t \in T(x^q)$. Observe that this point belongs to O^{ε}_{β}. In both cases, we have guaranteed the descent property because the Weiszfeld algorithm provides a descent sequence (whenever $\nabla F_{\varepsilon,\beta}(x^q) \neq 0$) and the function $F_{\varepsilon,\beta}(\cdot)$ is convex. Notice that by similar arguments as the ones in the proof of Lemma 4.4 we have that $\hat{\Phi}_{\beta}(x^q) = x^q$ if and only if $x^q = x^*$, where x^* denotes the optimal solution of Problem (2).

Example 4.2. Consider Example 2.1, which means, $a_1 = (4,1)$ and $a_2 = (1,0)$ are the demand points, $p = 1.5$, $\omega_1 = \omega_2 = 1$ and $\lambda_1 = 1.11$, $\lambda_2 = 1.12$. For $\varepsilon = 0.001$, $x^q = (2.65, 0.3) \in \mathrm{int}(O^{\varepsilon}_{(1,2)})$ does not satisfy condition (6),

$$\frac{\omega_2 \|h_{\varepsilon}(x - a_2)\|_p - \omega_1 \|h_{\varepsilon}(x - a_1)\|_p}{\omega_1 + \omega_2} - 2^{1/p}\varepsilon^{1/2} = -0.02 < 0.$$

Using (9), we get that $\hat{\Phi}_{(1,2)}(x^q) = (2.58, 0.40) \in \mathrm{int}(O^{\varepsilon}_{(1,2)})$. Therefore, $x^{q+1} = \hat{\Phi}_{(1,2)}(x^q)$ and $F_{\varepsilon}(x^{q+1}) = 3.77 < 3.79 = F_{\varepsilon}(x^q)$ (see Fig. 2).

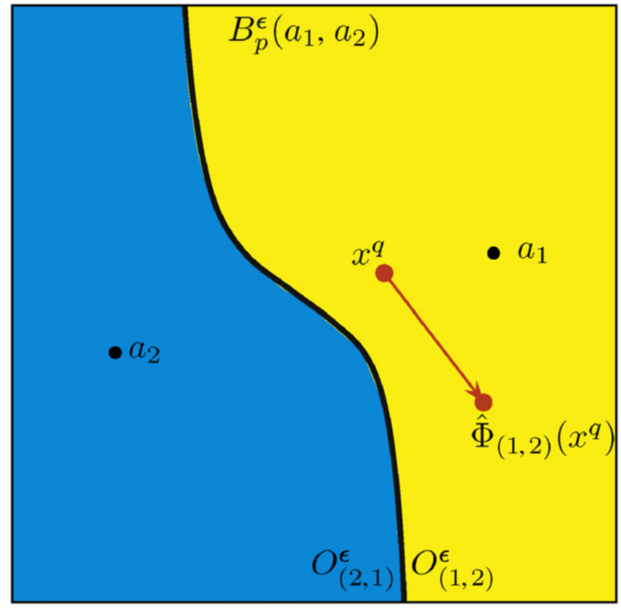

Fig. 2. $\hat{\Phi}_{(1,2)}(x^q) \in O^{\varepsilon}_{(1,2)}$.

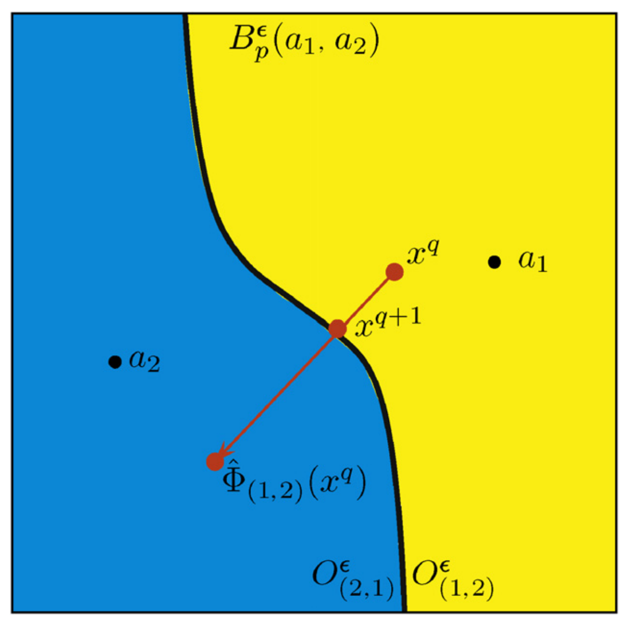

Fig. 3. $\hat{\Phi}_{(1,2)}(x^q) \notin O^{\varepsilon}_{(1,2)}$.

We now consider the above example but $\lambda_1 = 1$ and $\lambda_2 = 2$. In this case, using (9), we get that $\hat{\Phi}_{(1,2)}(x^q) = (2.08, 0.25)$, that is, $\hat{\Phi}_{(1,2)}(x^q) \in \mathrm{int}(O^{\varepsilon}_{(2,1)})$ and $F_{\varepsilon}(x^q) = 5.14 < 5.60 = F_{\varepsilon}(\hat{\Phi}_{(1,2)}(x^q))$. Hence, $x^{q+1} = (2.62, 0.30)$, the intersection point between $B^{\varepsilon}_p(a_1, a_2)$ and the segment $[x^q, \hat{\Phi}_{(1,2)}(x^q)]$ (see Fig. 3). Moreover, $F_{\varepsilon}(x^{q+1}) = 5.10 < 5.14 = F_{\varepsilon}(x^q)$.

4.2. *x^q belongs to the boundary of an ordered region*

In order to obtain a procedure that allows us to obtain a new point x^{q+1}, such that $F_{\varepsilon}(x^{q+1}) < F_{\varepsilon}(x^q)$, we first analyze the case where x^q belongs to only one bisector and then, we will study the general case where x^q belongs to more than one bisector.

4.2.1. x^q belongs to one bisector

In this case, the vector of weighted approximated distances from the demand points to x^q satisfies

(i) $\omega_{\beta_s}\|h_\varepsilon(x^q-a_{\beta_s})\|_p=\omega_{\beta_{s+1}}\|h_\varepsilon(x^q-a_{\beta_{s+1}})\|_p$, for some $s\in\{1,\dots,M-1\}$,

(ii) $\omega_{\beta_i}\|h_\varepsilon(x^q-a_{\beta_i})\|_p<\omega_{\beta_{i+1}}\|h_\varepsilon(x^q-a_{\beta_{i+1}})\|_p$, for any $i\neq s$, $i\in\{1,\dots,M-1\}$,

for some $\beta(s)=(\beta_1,\dots,\beta_M)\in\mathscr{P}(M)$. Equivalently, $x^q\in O^\varepsilon_{\beta(s)}\cap O^\varepsilon_{\beta'(s)}$, with $\beta'(s)=(\beta_1,\dots,\beta_{s-1},\beta_{s+1},\beta_s,\beta_{s+2}\dots,\beta_M)\in\mathscr{P}(M)$. Obviously, we have two different expressions of the ordered median objective function for x^q, as sum of the weighted approximated distances:

$$F_{\varepsilon,\beta(s)}(x^q)=\sum_{i\neq s,s+1}^{M}\lambda_i\omega_{\beta_i}\|h_\varepsilon(x^q-a_{\beta_i})\|_p+\lambda_s\omega_{\beta_s}\|h_\varepsilon(x^q-a_{\beta_s})\|_p+\lambda_{s+1}\omega_{\beta_{s+1}}\|h_\varepsilon(x^q-a_{\beta_{s+1}})\|_p,$$

$$F_{\varepsilon,\beta'(s)}(x^q)=\sum_{i\neq s,s+1}^{M}\lambda_i\omega_{\beta_i}\|h_\varepsilon(x^q-a_{\beta_i})\|_p+\lambda_s\omega_{\beta_{s+1}}\|h_\varepsilon(x^q-a_{\beta_{s+1}})\|_p+\lambda_{s+1}\omega_{\beta_s}\|h_\varepsilon(x^q-a_{\beta_s})\|_p,$$

with $F_{\varepsilon,\beta(s)}(x^q)=F_{\varepsilon,\beta'(s)}(x^q)$.

Now we analyze the case where x^q satisfies condition (6) for all $t\neq s$, $t\in\{1,\dots,M-1\}$. After that, we will study the case where some $t\neq s$ exists for which x^q does not satisfy (6).

Condition (6) *is fulfilled*:

In order to give an expression of x^{q+1} we define $\Phi_{\beta(s)}(x^q)=(\Phi_{\beta(s),1}(x^q),\Phi_{\beta(s),2}(x^q))$ as

$$\Phi_{\beta(s),j}(x^q)=x^q_j-K^s(x^q)\cdot D^{\beta(s)}_j(x^q),\quad j=1,2,$$

where $D^{\beta(s)}_j(x^q)$ was defined in (5) and

$$K^s(x)=\min\left\{1,\frac{1}{\|\eta(x)\|_p}\times\min_{t\neq s}\left\{\frac{\omega_{\beta_{t+1}}\|h_\varepsilon(x-a_{\beta_{t+1}})\|_p-\omega_{\beta_t}\|h_\varepsilon(x-a_{\beta_t})\|_p}{\omega_{\beta_{t+1}}+\omega_{\beta_t}}-2^{1/p}\varepsilon^{1/2}\right\}\right\},\quad(10)$$

with $\eta(x)=(\eta_1(x),\eta_2(x))$ such that

$$\eta_j(x)=\max\{|D^{\beta(s)}_j(x^q)|,|D^{\beta'(s)}_j(x^q)|\},\quad j=1,2.\quad(11)$$

If $\Phi_{\beta(s)}(x^q)\in O^\varepsilon_{\beta(s)}$ we define $x^{q+1}=\Phi_{\beta(s)}(x^q)$. If $\Phi_{\beta(s)}(x^q)\notin O^\varepsilon_{\beta(s)}$ and $\Phi_{\beta'(s)}(x^q)\in O^\varepsilon_{\beta'(s)}$, then we define $x^{q+1}=\Phi_{\beta'(s)}(x^q)$. In order to give an expression of x^{q+1} when $\Phi_{\beta(s)}(x^q)\notin O^\varepsilon_{\beta(s)}$ and $\Phi_{\beta'(s)}(x^q)\notin O^\varepsilon_{\beta'(s)}$ we define the function $G^{\alpha,s}_\varepsilon:\mathbb{R}^2\to\mathbb{R}$ as follows:

$$G^{\alpha,s}_\varepsilon(x)=(1-\alpha)F_{\varepsilon,\beta(s)}(x)+\alpha F_{\varepsilon,\beta'(s)}(x).$$

Notice that $G^{\alpha,s}_\varepsilon(x)=F_\varepsilon(x)$ for any $x\in B^\varepsilon_p(a_{\beta_s},a_{\beta_{s+1}})\cap(O^\varepsilon_{\beta(s)}\cup O^\varepsilon_{\beta'(s)})$ and $\alpha\in[0,1]$. Taking this into account, we define the following algorithmic map $\Psi^{\alpha^*,s}(x)=(\Psi^{\alpha^*,s}_1(x),\Psi^{\alpha^*,s}_2(x))$, with iterates given by $x^{q+1}=\Psi^{\alpha^*,s}(x^q)$, where

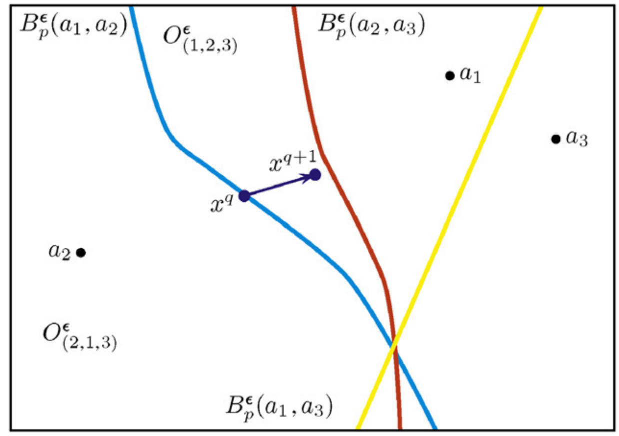

Fig. 4. $\Phi_{(1,2,3)}(x^q)\in O^\varepsilon_{(1,2,3)}$.

$$\Psi^{\alpha^*,s}_j(x^q)=x^q_j-K^s(x^q)\mathscr{D}^s_{\alpha^*,j}(x^q),\quad j=1,2,$$

with $K^s(x^q)$ defined by (10),

$$\mathscr{D}^s_{\alpha^*,j}(x^q):=[(1-\alpha^*)C_{\beta(s),j}(x^q)+\alpha^*C_{\beta'(s),j}(x^q)]^{-1}\times[(1-\alpha^*)\nabla_jF_{\varepsilon,\beta(s)}(x^q)+\alpha^*\nabla_jF_{\varepsilon,\beta'(s)}(x^q)],\quad j=1,2,\quad(12)$$

and $\alpha^*\in(0,1)$ such that $\Psi^{\alpha^*,s}(x^q)\in B^\varepsilon_p(a_{\beta_s},a_{\beta_{s+1}})\cap(O^\varepsilon_{\beta(s)}\cup O^\varepsilon_{\beta'(s)})$. The existence of such $\alpha^*\in(0,1)$ is proven by Lemmas A.1 and A.2 in Appendix A.

Therefore, if we define the function

$$\xi_{x^q}(\alpha):=\omega_{\beta_s}\|h_\varepsilon(\Psi^{\alpha,s}(x^q)-a_{\beta_s})\|_p-\omega_{\beta_{s+1}}\|h_\varepsilon(\Psi^{\alpha,s}(x^q)-a_{\beta_{s+1}})\|_p,\quad(13)$$

then α^* satisfies that $\xi_{x^q}(\alpha^*)=0$.

We summarize the iterative scheme when $x^q\in B^\varepsilon_p(a_{\beta_s},a_{\beta_{s+1}})$ and condition (6) is fulfilled:

$$x^{q+1}=\begin{cases}\Phi_{\beta(s)}(x^q) & \text{if } \Phi_{\beta(s)}(x^q)\in O^\varepsilon_{\beta(s)},\\ \Phi_{\beta'(s)}(x^q) & \text{if } \Phi_{\beta(s)}(x^q)\notin O^\varepsilon_{\beta(s)} \text{ and } \Phi_{\beta'(s)}(x^q)\in O^\varepsilon_{\beta'(s)},\\ \Psi^{\alpha^*,s}(x^q) & \text{otherwise.}\end{cases}\quad(14)$$

Example 4.3. Consider three demand points, $a_1=(4,1)$, $a_2=(1,0)$ and $a_3=(4.5,0.5)$, with $\omega_1=\omega_2=\omega_3=1$ and $\lambda_1=1$, $\lambda_2=2$ and $\lambda_3=3$. For $p=1.5$, $\varepsilon=0.001$ and $x^q=(2.62,0.30)$, we get that $x^q\in B^\varepsilon_p(a_1,a_2)$. We can check that $\Phi_{(1,2,3)}(x^q)=(2.65,0.30)\in\mathrm{int}(O^\varepsilon_{(1,2,3)})$. Therefore, $x^{q+1}=\Phi_{(1,2,3)}(x^q)$ and $F_\varepsilon(x^{q+1})=10.82<10.88=F_\varepsilon(x^q)$ (see Fig. 4).

On the other hand, continuing with Example 4.2, for $x^q=(2.62,0.30)$ and $\varepsilon=0.001$, we have that $x^q\in B^\varepsilon_p(a_1,a_2)$. For $\lambda_1=1$ and $\lambda_2=2$, we can check that $\Phi_{(1,2)}(x^q)=(2.05,0.25)\in\mathrm{int}(O^\varepsilon_{(2,1)})$ and $\Phi_{(2,1)}(x^q)=(3.05,0.57)\in\mathrm{int}(O^\varepsilon_{(1,2)})$. In addition, $F_\varepsilon(x^q)=5.10<5.63=F_\varepsilon(\Phi_{(1,2)}(x^q))$ and $F_\varepsilon(x^q)=5.10<5.63=F_\varepsilon(\Phi_{(2,1)}(x^q))$. Hence, using (14) we have that $x^{q+1}=\Psi^{\alpha^*,1}(x^q)=(2.56,0.39)$ (see Fig. 5). Moreover, $F_\varepsilon(x^{q+1})=5.07<5.10=F_\varepsilon(x^q)$.

Lemma 4.5. *The function $\psi^{\alpha^*,s}(\cdot)$ is continuous in a neighborhood of x^q, provided that $\partial\xi_{x^q}(\alpha^*)/\partial\alpha\neq0$, where ξ is defined by* (13).

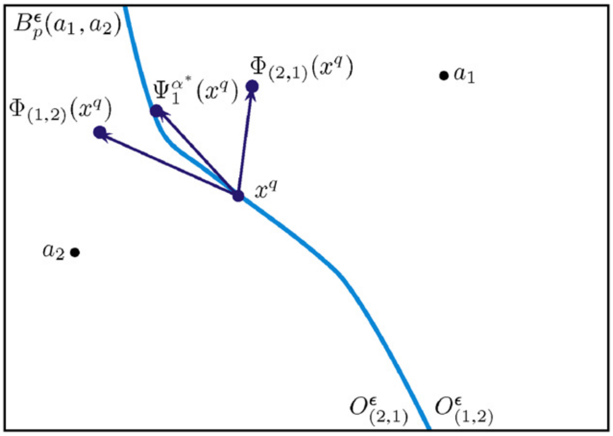

Fig. 5. $\Phi_{(1,2)}(x^q) \in O^{\varepsilon}_{(2,1)}$, $\Phi_{(2,1)}(x^q) \in O^{\varepsilon}_{(1,2)}$.

Proof. The function $\psi^{\alpha^*,s}(x^q)$ depends on α^*, where α^* satisfies that $\xi_{x^q}(\alpha^*)=0$ (that is, α^* also depends on x^q). We prove the continuity of $\psi^{\alpha^*,s}(\cdot)$ using the Implicit Function Theorem.

By definition, the function $\xi_x(\alpha)$ is continuous and differentiable with respect to x, and $\partial\xi_{x^q}(\alpha^*)/\partial\alpha \neq 0$. Since the hypotheses of the Implicit Function Theorem are satisfied, a continuous function, $H(\cdot)$, exists in a neighborhood of x^q, $N(x^q)$, such that $\alpha = H(x)$ and $\xi_x(H(x)) = 0$, $\forall x \in N(x^q)$, and the result follows. $\square$

Lemma 4.6. *Let $x^q \in B^{\varepsilon}_p(a_{\beta_s}, a_{\beta_{s+1}})$ and $x^q \notin B^{\varepsilon}_p(a_{\beta_t}, a_{\beta_{t+1}})$, $\forall t \neq s$. If $x^{q+1} \neq x^q$, where x^{q+1} is given by* (14), *then $F_{\varepsilon}(x^{q+1}) < F_{\varepsilon}(x^q)$, $\forall q \geqslant 1$.*

Proof. Let $x^{q+1} \neq x^q$ be given by (14). If either $x^{q+1} = \Phi_{\beta(s)}(x^q)$ or $x^{q+1} = \Phi_{\beta'(s)}(x^q)$, we can prove that $F_{\varepsilon}(x^{q+1}) < F_{\varepsilon}(x^q)$ by using similar arguments as the ones in Lemma 4.3. We analyze the case where $x^{q+1} = \Psi^{\alpha^*,s}(x^q)$. Since x^q and $\Psi^{\alpha^*,s}(x^q)$ belongs to $B^{\varepsilon}_p(a_{\beta_s}, a_{\beta_{s+1}}) \cap (O^{\varepsilon}_{\beta(s)} \cup O^{\varepsilon}_{\beta'(s)})$, we have that

$$F_{\varepsilon}(x^q) = G^{\alpha^*,s}_{\varepsilon}(x^q) \quad \text{and} \quad F_{\varepsilon}(\Psi^{\alpha^*,s}(x^q)) = G^{\alpha^*,s}_{\varepsilon}(\Psi^{\alpha^*,s}(x^q)).$$

In addition, $G^{\alpha^*,s}_{\varepsilon}(x^q)$ can be rewritten like a sum of weighted distances as follows:

$$G^{\alpha^*,s}_{\varepsilon}(x) = \sum_{i=1}^{M} \gamma_i \|h_{\varepsilon}(x - a_{\beta_i})\|_p,$$

where $\gamma_s = \alpha^*(\lambda_s + \lambda_{s+1})\omega_{\beta_s}$, $\gamma_{s+1} = (1-\alpha^*)(\lambda_s + \lambda_{s+1})\omega_{\beta_{s+1}}$ and $\gamma_i = \lambda_i \omega_{\beta_i}$, for $i = 1, \ldots, M$, with $i \neq s, s+1$. Hence, $\Psi^{\alpha^*,s}(x^q)$ is given by the map defining the approximated Weiszfeld algorithm with smaller stepsize applied to a sum of weighted approximated distances, $G^{\alpha^*,s}_{\varepsilon}$, at x^q (see [20]). Thus, $G^{\alpha^*,s}_{\varepsilon}(\Psi^{\alpha^*,s}(x^q)) < G^{\alpha^*,s}_{\varepsilon}(x^q)$ and the result follows. $\square$

Lemma 4.7. *Let $x^q \in B^{\varepsilon}_p(a_{\beta_s}, a_{\beta_{s+1}})$ and denote x^* the optimal solution of Problem* (2). *If x^{q+1} is given by* (14), *we have that*:

(i) *If $x^{q+1} = x^q$ then $x^q = x^*$.*
(ii) *If $x^q = x^*$ then $x^{q+1} = x^*$.*

Proof. Statements (i) and (ii) when $x^{q+1} = \Phi_{\beta(s)}(x^q)$ or $x^{q+1} = \Phi_{\beta'(s)}(x^q)$ can be proven analogously to Lemma 4.4. Therefore we focus the proof on the case where $\Phi_{\beta'(s)}(x^q) \notin O^{\varepsilon}_{\beta'(s)}$, $\Phi_{\beta(s)}(x^q) \notin O^{\varepsilon}_{\beta(s)}$ and $\Psi^{\alpha^*,s}(x^q) = x^q$.

First, we prove that $F_{\varepsilon}(x^q) < F_{\varepsilon}(y)$ for any $y (\neq x^q) \in B^{\varepsilon}_p(a_{\beta_s}, a_{\beta_{s+1}}) \cap (O^{\varepsilon}_{\beta(s)} \cup O^{\varepsilon}_{\beta'(s)})$. Indeed, using that $\Psi^{\alpha^*,s}(x^q) = x^q$, by the definition of $\Psi^{\alpha^*,s}(x^q)$, we have that $\nabla G^{\alpha^*,s}_{\varepsilon}(x^q) = (0,0)$. In addition, since $G^{\alpha^*,s}_{\varepsilon}(\cdot)$ is a strictly convex function (recall that $G^{\alpha,s}_{\varepsilon}(\cdot)$ is the weighted sum of approximated distances), we have that x^q is the minimum of $G^{\alpha^*,s}_{\varepsilon}(\cdot)$.

Moreover, $G^{\alpha^*,s}_{\varepsilon}(y) = F_{\varepsilon}(y)$ for any $y \in B^{\varepsilon}_p(a_{\beta_s}, a_{\beta_{s+1}}) \cap (O^{\varepsilon}_{\beta(s)} \cup O^{\varepsilon}_{\beta'(s)})$. Therefore, $F_{\varepsilon}(x^q) = G^{\alpha^*,s}_{\varepsilon}(x^q) < G^{\alpha^*,s}_{\varepsilon}(y) = F_{\varepsilon}(y)$ for any $y (\neq x^q) \in B^{\varepsilon}_p(a_{\beta_s}, a_{\beta_{s+1}}) \cap (O^{\varepsilon}_{\beta(s)} \cup O^{\varepsilon}_{\beta'(s)})$.

Second, we prove that $F_{\varepsilon,\beta(s)}(x^q) < F_{\varepsilon,\beta(s)}(y)$, for all $y (\neq x^q) \in O^{\varepsilon}_{\beta(s)}$. Indeed, we have that: (1) the function $F_{\varepsilon,\beta(s)}(y)$ increases when y moves from x^q along $B^{\varepsilon}_p(a_{\beta_s}, a_{\beta_{s+1}})$ within $(O^{\varepsilon}_{\beta(s)} \cup O^{\varepsilon}_{\beta'(s)})$ and (2) $\Phi_{\beta(s)}(x^q) \notin O^{\varepsilon}_{\beta(s)}$. Therefore, by convexity arguments, no direction of decrease exists in $O^{\varepsilon}_{\beta(s)}$, that is, $F_{\varepsilon,\beta(s)}(x^q) < F_{\varepsilon,\beta(s)}(y)$ for any $y (\neq x^q) \in O^{\varepsilon}_{\beta(s)}$. Analogously, we can obtain that $F_{\varepsilon,\beta'(s)}(x^q) < F_{\varepsilon,\beta'(s)}(y)$ for all $y (\neq x^q) \in O^{\varepsilon}_{\beta'(s)}$ and the result follows.

For the statement (ii), if $\Psi^{\alpha^*,s}(x^q) = x^{q+1} \neq x^q$ we have proved that $F_{\varepsilon}(x^q) > F_{\varepsilon}(x^{q+1})$, contradicting that x^* is the optimal solution. $\square$

Condition (6) *is not fulfilled*:

Let $x^q \in B^{\varepsilon}_p(a_{\beta_s}, a_{\beta_{s+1}})$ be such that it does not satisfy condition (6) for some $t \in \{1, \ldots, M-1\}$, $t \neq s$, and $T(x^q)$ given by (8) (notice that $s \in T(x^q)$). We define $\hat{\Phi}_{\beta(s)}(x^q) = (\hat{\Phi}_{\beta(s),1}(x^q), \hat{\Phi}_{\beta(s),2}(x^q))$ as

$$\hat{\Phi}_{\beta(s),j}(x^q) = x^q_j - \hat{K}^s(x^q) \cdot D^{\beta(s)}_j(x^q), \quad j = 1,2, \tag{15}$$

where $D^{\beta(s)}_j(x^q)$ was defined in (5) and

$$\hat{K}^s(x) = \min\left\{1, \frac{1}{\|\eta(x)\|_p} \times \min_{t \notin T(x^q)} \left\{ \frac{\omega_{\beta_{t+1}} \|h_{\varepsilon}(x - a_{\beta_{t+1}})\|_p - \omega_{\beta_t} \|h_{\varepsilon}(x - a_{\beta_t})\|_p}{\omega_{\beta_{t+1}} + \omega_{\beta_t}} - 2^{1/p}\varepsilon^{1/2} \right\}\right\},$$

and $\eta(x)$ is given by (11).

Analogously, we define $\hat{\Phi}_{\beta'(s)}(x^q)$. If $\hat{\Phi}_{\beta(s)}(x^q) \in O^{\varepsilon}_{\beta(s)}$, we define $x^{q+1} = \hat{\Phi}_{\beta(s)}(x^q)$. If $\hat{\Phi}_{\beta(s)}(x^q) \notin O^{\varepsilon}_{\beta(s)}$ and $\hat{\Phi}_{\beta'(s)}(x^q) \in O^{\varepsilon}_{\beta'(s)}$, then we define $x^{q+1} = \hat{\Phi}_{\beta'(s)}(x^q)$. In order to give an expression of x^{q+1} when $\hat{\Phi}_{\beta(s)}(x^q) \notin O^{\varepsilon}_{\beta(s)}$ and $\hat{\Phi}_{\beta'(s)}(x^q) \notin O^{\varepsilon}_{\beta'(s)}$, we define the following map $\hat{\Psi}^{\alpha^*,s}(x) = (\hat{\Psi}^{\alpha^*,s}_1(x), \hat{\Psi}^{\alpha^*,s}_2(x))$, as

$$\hat{\Psi}^{\alpha^*,s}_j(x^q) = x^q_j - \hat{K}^s(x^q)\mathscr{D}^s_{\alpha^*,j}(x^q), \quad j = 1,2, \tag{16}$$

where $\mathscr{D}^s_{\alpha^*,j}(x^q)$ was given by (12) and α^* satisfies that $\xi_{x^q}(\alpha^*) = 0$, (see (13)). From Lemma A.2 in Appendix A, we have that $\hat{\Psi}^{\alpha^*,s}(x^q) \in B^{\varepsilon}_p(a_{\beta_s}, a_{\beta_{s+1}})$ but we have not guaranteed that $\hat{\Psi}^{\alpha^*,s}(x^q) \in O^{\varepsilon}_{\beta(s)} \cup O^{\varepsilon}_{\beta'(s)}$. If $\hat{\Psi}^{\alpha^*,s}(x^q) \in O^{\varepsilon}_{\beta(s)} \cup O^{\varepsilon}_{\beta'(s)}$, we define $x^{q+1} = \hat{\Psi}^{\alpha^*,s}(x^q)$. If $\hat{\Psi}^{\alpha^*,s}(x^q) \notin O^{\varepsilon}_{\beta(s)} \cup O^{\varepsilon}_{\beta'(s)}$, we define $x^{q+1} = x^q_{\alpha^*}$, where $x^q_{\alpha^*}$ is the intersection point closest to x^q of the segment $[x^q, \hat{\Psi}^{\alpha^*,s}(x^q)]$ and a bisector $B_p^\varepsilon(a_t, a_{t+1})$, for some $t(\neq s) \in T(x^q)$.

We summarize the iterative scheme when $x^q \in B_p^\varepsilon(a_{\beta_s}, a_{\beta_{s+1}})$ and condition (6) is not fulfilled:

$$x^{q+1} = \begin{cases} \hat{\Phi}_{\beta(s)}(x^q) & \text{if } \hat{\Phi}_{\beta(s)}(x^q) \in O^\varepsilon_{\beta(s)}, \\ \hat{\Phi}_{\beta'(s)}(x^q) & \text{if } \hat{\Phi}_{\beta(s)}(x^q) \notin O^\varepsilon_{\beta(s)} \\ & \quad \text{and } \hat{\Phi}_{\beta'(s)}(x^q) \in O^\varepsilon_{\beta'(s)}, \\ \hat{\Psi}^{\alpha^*,s}(x^q) & \text{if } \hat{\Phi}_{\beta(s)}(x^q) \notin O^\varepsilon_{\beta(s)}, \hat{\Phi}_{\beta'(s)}(x^q) \notin O^\varepsilon_{\beta'(s)} \\ & \quad \text{and } \hat{\Psi}^{\alpha^*,s}(x^q) \in O^\varepsilon_{\beta(s)} \cup O^\varepsilon_{\beta'(s)}, \\ x^q_{\alpha^*} & \text{otherwise.} \end{cases} \tag{17}$$

In the first three cases, since x^{q+1} can be obtained by a modified gradient method with stepsize lower than the Weiszfeld algorithm (analogously to Lemma 4.7), we have that $F_\varepsilon(x^{q+1}) < F_\varepsilon(x^q)$. For the last case, we now prove the descent property. Let $x^{q+1} = x^q_{\alpha^*}$. If $x^q_{\alpha^*} \in O^\varepsilon_{\beta(s)}$, by the descent property of Weiszfeld algorithm we have that $F_{\varepsilon,\beta(s)}(x^q) = G_\varepsilon^{\alpha^*,s}(x^q) > G_\varepsilon^{\alpha^*,s}(\hat{\Psi}^{\alpha^*,s}(x^q)) = F_{\varepsilon,\beta(s)}(\hat{\Psi}^{\alpha^*,s}(x^q))$. Hence, since $x^q_{\alpha^*}$ belongs to the segment defined by x^q and $\hat{\Psi}^{\alpha^*,s}(x^q)$, using the strict convexity of $F_{\varepsilon,\beta(s)}(\cdot)$, we obtain that $F_{\varepsilon,\beta(s)}(x^q) > F_{\varepsilon,\beta(s)}(x^q_{\alpha^*})$. Therefore, since $x^q, x^q_{\alpha^*} \in O^\varepsilon_{\beta(s)}$ we have that $F_\varepsilon(x^q) > F_\varepsilon(x^q_{\alpha^*})$. Analogously, we can obtain that $F_\varepsilon(x^q) > F_\varepsilon(x^q_{\alpha^*})$ when $x^q_{\alpha^*} \in O^\varepsilon_{\beta'(s)}$.

Moreover, by using similar arguments as the ones in the proof of Lemmas 4.4 and 4.7, we have that x^q is the optimal solution of Problem (2) when $\hat{\Phi}_{\beta(s)}(x^q) = x^q$, $\hat{\Phi}_{\beta'(s)}(x^q) = x^q$ or $\hat{\Psi}^{\alpha^*,s}(x^q) = x^q$. Observe that under conditions of Lemma 4.5 the function $\hat{\Psi}^{\alpha^*,s}(\cdot)$ is also continuous in a neighborhood of x^q.

Example 4.4. Consider three demand points $a_1 = (4, 1)$, $a_2 = (1, 0)$ and $a_3 = (3, -0.5)$, $p = 1.5$, $\omega_1 = \omega_2 = \omega_3 = 1$ and $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 5$. For $x^q = (2.28, 0.9)$ and $\varepsilon = 0.001$, we have that $x^q \in B_p^\varepsilon(a_1, a_2)$ and $x^q \in O^\varepsilon_{(3,1,2)} \cap O^\varepsilon_{(3,2,1)}$. In addition, x^q satisfies condition (6) neither for a_2 and a_3 nor for a_1 and a_3. Using (15), we can verify that $\hat{\Phi}_{(3,1,2)}(x^q) = (1.97, 0.47) \notin O^\varepsilon_{(3,1,2)}$ and $\hat{\Phi}_{(3,2,1)}(x^q) = (3.04, 0.82) \notin O^\varepsilon_{(3,2,1)}$. In this case, using (16) we get that $x^{q+1} = \hat{\Psi}^{\alpha^*,1}(x^q) = (2.41, 0.66) \in O^\varepsilon_{(3,1,2)} \cup O^\varepsilon_{(3,2,1)}$ (see Fig. 6). Moreover, $F_\varepsilon(x^{q+1}) = 13.28 < 13.91 = F_\varepsilon(x^q)$.

Consider now the above example but $\lambda_1 = 1.1$, $\lambda_2 = 1.2$, $\lambda_3 = 1.3$. For $x^q = (2.23, 1)$ and $\varepsilon = 0.001$, we have that $x^q \in B_p^\varepsilon(a_1, a_2)$ and $x^q \in O^\varepsilon_{(1,2,3)} \cap O^\varepsilon_{(2,1,3)}$. Moreover, x^q satisfies condition (6) neither for a_2 and a_3 nor for a_1 and a_3. Using (15), we can verify that $\hat{\Phi}_{(1,2,3)}(x^q) = (2.60, 0.93) \notin O^\varepsilon_{(1,2,3)}$ and $\hat{\Phi}_{(2,1,3)}(x^q) = (2.68, 0.93) \notin O^\varepsilon_{(2,1,3)}$. Hence, using (16) we have that $\hat{\Psi}^{\alpha^*,1}(x^q) = (2.29, 0.87) \in O^\varepsilon_{(3,1,2)}$, that is, $\hat{\Psi}^{\alpha^*,1}(x^q) \notin O^\varepsilon_{(1,2,3)} \cup O^\varepsilon_{(2,1,3)}$. Thus, $x^{q+1} = (2.27, 0.91)$, where $(2.27, 0.91)$ is the intersection point between the segment $[x^q, \hat{\Psi}^{\alpha^*,1}(x^q)]$ and bisector $B_p^\varepsilon(a_2, a_3)$ (see Fig. 7). Moreover, $F_\varepsilon(x^{q+1}) = 6.28 < 6.48 = F_\varepsilon(x^q)$.

4.2.2. x^q belongs to more than one bisector

Finally, we study the case where x^q belongs to k bisector lines with $k > 1$. For this case, the procedures developed in the previous section may not give a point x^{q+1} with lower objective value. Moreover, since x^q belongs to more than one bisector, we may have to deal simultaneously with many different ordered regions and the search for a point x^{q+1} with lower objective function would be a very difficult task. Therefore, in order to simplify the procedure, we will first verify if x^q is an optimal solution for Problem (2).

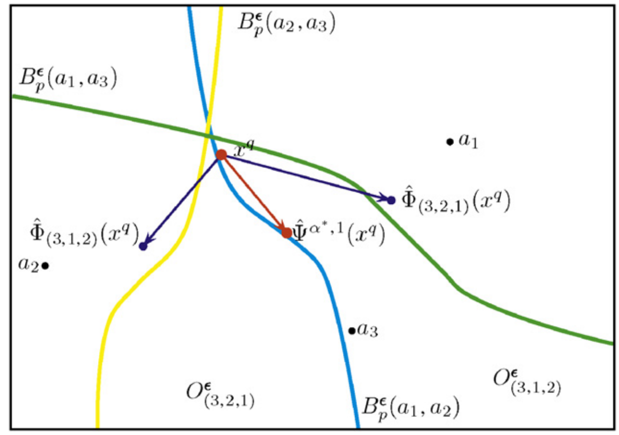

Fig. 6. $\hat{\Psi}^{\alpha^*,1}(x^q) \in O^\varepsilon_{(3,1,2)} \cup O^\varepsilon_{(3,2,1)}$.

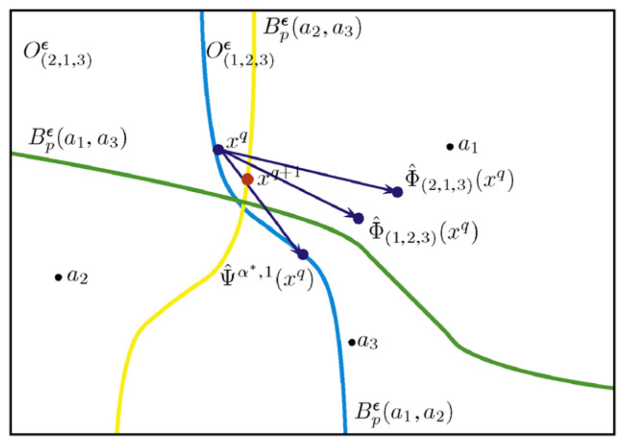

Fig. 7. $\hat{\Psi}^{\alpha^*,1}(x^q) \notin O^\varepsilon_{(1,2,3)} \cup O^\varepsilon_{(2,1,3)}$.

In order to check the optimality of x^q, we compute $\partial F_\varepsilon(x^q)$ which is given by (see [29])

$$\partial F_\varepsilon(x^q) = co\{\cup \nabla F_{\varepsilon,\beta}(x^q) : \beta \in \Sigma(x^q)\},$$

where $\Sigma(x^q) := \{\beta \in \mathscr{P}(M) : F_{\varepsilon,\beta}(x^q) = F_\varepsilon(x^q)\}$.

Therefore, x^q is the optimal solution of Problem (2) if and only if $(0, 0) \in \mathrm{co}(\bigcup_{\beta \in \Sigma(x^q)} \nabla F_{\varepsilon,\beta}(x^q))$.

In the following, we analyze the case when x^q is not the optimal solution. Define the set $S(x^q)$ as follows:

$$S(x^q) = \{s \in \{1, \ldots, M-1\} : \omega_{\beta_s} \|h_\varepsilon(x^q - a_{\beta_s})\|_p = \omega_{\beta_{s+1}} \|h_\varepsilon(x^q - a_{\beta_{s+1}})\|_p\}.$$

We will distinguish between x^q satisfying condition (6) for all $t \notin S(x^q)$ and x^q not satisfying (6), for some $t \notin S(x^q)$.

Condition (6) *is fulfilled*:

For each $\beta \in \Sigma(x^q)$, we define $\Phi^S_\beta(x^q) = (\Phi^S_{\beta,1}(x^q), \Phi^S_{\beta,2}(x^q))$ as

$$\Phi^S_{\beta,j}(x^q) = x^q_j - K^S(x^q) \cdot D^\beta_j(x^q), \quad j = 1, 2, \tag{18}$$

where $D_j^{\beta}(x^q)$ was defined in (5),

$$K^S(x^q) = \min\left\{1, \frac{1}{\|\eta^S(x^q)\|_p} \times \min_{t \notin S(x^q)}\left\{\frac{\omega_{\beta_{t+1}}\|h_\varepsilon(x^q - a_{\beta_{t+1}})\|_p - \omega_{\beta_t}\|h_\varepsilon(x^q - a_{\beta_t})\|_p}{\omega_{\beta_{t+1}}+\omega_{\beta_t}} - 2^{1/p}\varepsilon^{1/2}\right\}\right\},$$

and $\eta^S(x^q) = (\eta_1^S(x^q), \eta_2^S(x^q))$ is defined by

$$\eta_j^S(x^q) = \max_{\beta \in \Sigma(x^q)} \{|C_{\beta j}^{-1}(x^q)\nabla_j F_{\varepsilon,\beta}(x^q)|\}, \quad j = 1, 2. \tag{19}$$

If some $\beta \in \Sigma(x^q)$ exists such that $\Phi_\beta^S(x^q) \in O_\beta^\varepsilon$, we define $x^{q+1} = \Phi_\beta^S(x^q)$, where β is the first found permutation with that condition. If $\Phi_\beta^S(x^q) \notin O_\beta^\varepsilon$, for all $\beta \in \Sigma(x^q)$, in order to give an expression of x^{q+1}, we consider a sufficiently small neighborhood of x^q, $N(x^q)$. If $\beta \in \Sigma(x^q)$ exists such that the intersection point between the boundary of $N(x^q)$ and the segment $[x^q, \Phi_\beta^S(x^q)]$ belongs to O_β^ε, we define x^{q+1} as this intersection point, denoted by x_β^q. Otherwise, since x^q is not an optimal solution, there exists a decrease direction in $B_p^\varepsilon(a_s, a_{s+1})$, for some $s \in S(x^q)$. We compare the objective value for each of these intersection points between the boundary of $N(x^q)$ and $B_p^\varepsilon(a_s, a_{s+1})$, for any $s \in S(x^q)$. Define x^{q+1} as the intersection point with the lowest objective value, denoted by x_s^q.

Condition (6) *is not fulfilled*:

Let $T(x^q)$ defined as (8). Notice that $S(x^q) \subset T(x^q)$. For each $\beta \in \Sigma(x^q)$, we define the map $\hat{\Phi}_\beta^S(x^q) = (\hat{\Phi}_{\beta,1}^S(x^q), \hat{\Phi}_{\beta,2}^S(x^q))$ as

$$\hat{\Phi}_{\beta j}^S(x^q) = x_j^q - \hat{K}^S(x^q) \cdot D_j^\beta(x^q), \quad j = 1, 2, \tag{20}$$

where $D_j^\beta(x^q)$ was defined in (5),

$$\hat{K}^S(x^q) = \min\left\{1, \frac{1}{\|\eta^S(x^q)\|_p} \times \min_{t \notin \mathrm{T}(x^q)}\left\{\frac{\omega_{\beta_{t+1}}\|h_\varepsilon(x - a_{\beta_{t+1}})\|_p - \omega_{\beta_t}\|h_\varepsilon(x - a_{\beta_t})\|_p}{\omega_{\beta_{t+1}}+\omega_{\beta_t}} - 2^{1/p}\varepsilon^{1/2}\right\}\right\},$$

and $\eta(x^q)$ is given by (19). If there exists some $\beta \in \Sigma(x^q)$ such that $\hat{\Phi}_\beta^S(x^q) \in O_\beta^\varepsilon$, we define $x^{q+1} = \hat{\Phi}_\beta^S(x^q)$. If $\hat{\Phi}_\beta^S(x^q) \notin O_\beta^\varepsilon$, for all $\beta \in \Sigma(x^q)$, we proceed like in the case satisfying (6).

Notice that by using similar arguments as the ones in the proof of Lemmas 4.3 and 4.4, we have that $F_\varepsilon(\Phi_\beta^S(x^q)) < F_\varepsilon(x^q)$ ($F_\varepsilon(\hat{\Phi}_\beta^S(x^q)) < F_\varepsilon(x^q)$) and x^q coincides with the optimal solution of Problem (2) if and only if $\Phi_\beta^S(x^q) = x^q$ ($\hat{\Phi}_\beta^S(x^q) = x^q$), for some $\beta \in \Sigma(x^q)$.

5. The algorithm and properties

In this section we summarize the complete iterative scheme defined in the previous section and we also study the convergence of the proposed algorithm.

We give the expression of the algorithm depending on the relative position of the current iteration, x^q, with respect to the bisector lines and ordered regions. In order to do that, we define the algorithmic map $T_\varepsilon : x \to T_\varepsilon(x)$ as

$$x^{q+1} = T_\varepsilon(x^q) = \begin{cases} T_{\varepsilon,1}(x^q) & \text{if } |\Sigma(x^q)| = 1, \\ T_{\varepsilon,2}(x^q) & \text{if } |\Sigma(x^q)| = 2, \\ T_{\varepsilon,3}(x^q) & \text{if } |\Sigma(x^q)| > 2, \end{cases} \tag{21}$$

where

$$T_{\varepsilon,1}(x^q) = \begin{cases} \Phi_\beta(x^q) & \text{if } x^q \text{ satisfies (6)}, \\ & \quad \forall t \in \{1, \dots, M-1\} \text{ (see (7))}, \\ \hat{\Phi}_\beta(x^q) & \text{if } x^q \text{ does not satisfy (6), for some} \\ & \quad t \in \{1, \dots, M-1\} \text{ and } \hat{\Phi}_\beta(x^q) \in \mathrm{int}(O_\beta^\varepsilon) \\ & \quad \text{(see (9))}, \\ x_\beta^q & \text{otherwise (see the definition of } x_\beta^q \\ & \quad \text{in the paragraph above of} \\ & \quad \text{Example 4.2)}, \end{cases}$$

$$T_{\varepsilon,2}(x^q) = \begin{cases} T_{\varepsilon,2}^+(x^q) & \text{if } x^q \text{ satisfies (6), } \forall t \in \{1, \dots, M-1\}, \\ T_{\varepsilon,2}^-(x^q) & \text{otherwise,} \end{cases}$$

with

$$T_{\varepsilon,2}^+(x^q) = \begin{cases} \Phi_{\beta(s)}(x^q) & \text{if } \Phi_{\beta(s)}(x^q) \in O_{\beta(s)}^\varepsilon, \\ \Phi_{\beta'(s)}(x^q) & \text{if } \Phi_{\beta(s)}(x^q) \notin O_{\beta(s)}^\varepsilon \\ & \quad \text{and } \Phi_{\beta'(s)}(x^q) \in O_{\beta'(s)}^\varepsilon \text{ (see (14))}, \\ \Psi^{\alpha^*,s}(x^q) & \text{otherwise,} \end{cases}$$

$$T_{\varepsilon,2}^-(x^q) = \begin{cases} \hat{\Phi}_{\beta(s)}(x^q) & \text{if } \hat{\Phi}_{\beta(s)}(x^q) \in O_{\beta(s)}^\varepsilon, \\ \hat{\Phi}_{\beta'(s)}(x^q) & \text{if } \hat{\Phi}_{\beta(s)}(x^q) \notin O_{\beta(s)}^\varepsilon \\ & \quad \text{and } \hat{\Phi}_{\beta'(s)}(x^q) \in O_{\beta'(s)}^\varepsilon \text{ (see (17))}, \\ \hat{\Psi}^{\alpha^*,s}(x^q) & \text{if } \hat{\Phi}_{\beta(s)}(x^q) \notin O_{\beta(s)}^\varepsilon, \hat{\Phi}_{\beta'(s)}(x^q) \notin O_{\beta'(s)}^\varepsilon \\ & \quad \text{and } \hat{\Psi}^{\alpha^*,s}(x^q) \in O_{\beta(s)}^\varepsilon \cup O_{\beta'(s)}^\varepsilon, \\ x_{\alpha^*}^q & \text{otherwise} \end{cases}$$

and

$$T_{\varepsilon,3}(x^q) = \begin{cases} T_{\varepsilon,3}^+(x^q) & \text{if } x^q \text{ satisfies (6), } \forall t \in \{1, \dots, M-1\}, \\ T_{\varepsilon,3}^-(x^q) & \text{otherwise,} \end{cases}$$

where (see Section 4.2.2)

$$T_{\varepsilon,3}^+(x^q) = \begin{cases} \Phi_\beta^S(x^q) & \text{if } \Phi_\beta^S(x^q) \in O_\beta^\varepsilon \\ & \quad \text{for some } \beta \in \Sigma(x^q) \text{ (see (18))}, \\ x_\beta^q & \text{if } x_\beta^q \in O_\beta^\varepsilon \text{ for some } \beta \in \Sigma(x^q), \\ x_s^q & \text{otherwise,} \end{cases}$$

$$T_{\varepsilon,3}^-(x^q) = \begin{cases} \hat{\Phi}_\beta^S(x^q) & \text{if } \hat{\Phi}_\beta^S(x^q) \in O_\beta^\varepsilon \\ & \quad \text{for some } \beta \in \Sigma(x^q) \text{ (see (20))}, \\ \hat{x}_\beta^q & \text{if } x_\beta^q \in O_\beta^\varepsilon \text{ for some } \beta \in \Sigma(x^q), \\ \hat{x}_s^q & \text{otherwise.} \end{cases}$$

In the previous section, we have proved that the sequence $\{x^q\}_{q \in \mathbb{N}}$ generated by (21) verifies the descent property. Moreover, $x^{q+1} = x^q$ if and only if x^q coincides with the optimal solution of Problem (2). Taking into account these results, we give the following convergence result.

Theorem 5.1. *The sequence* $\{x^q\}_{q\in\mathbb{N}}$ *converges to the optimal solution of Problem* (2) *provided that* $\partial\xi_{x^q}(\alpha^*)/\partial\alpha\neq 0$, *where* ξ *was defined by* (13).

Proof. We can assume, without loss of generality, that $T_\varepsilon(x^q)\neq x^q$, for any q, where $T_\varepsilon(x^q)$ was defined in (21). Otherwise, we would have that the whole sequence converges to x^q in a finite number of steps.

Since the sequence $\{x^q\}_{q\in\mathbb{N}}$ is bounded (see Lemma A.3 in Appendix A), by the Bolzano–Weierstrass Theorem, at least one accumulation point exists. We will prove by contradiction that the sequence contains a unique accumulation point. Let us assume that the sequence $\{x^q\}_{q\in\mathbb{N}}$ has two accumulation points $p_1\neq p_2$ and consider a ball B_1, centered at p_1 such that $p_2\notin B_1$. Thus, we can choose a subsequence $\{x^{n_k}\}_{k\in\mathbb{N}}$ verifying:

(i) $\{x^{n_k}\}\to p_1$.
(ii) $x^{n_k+1}=T_\varepsilon(x^{n_k})\notin B_1$, for all $k\geqslant 1$.

Observe that this choice is possible because the non-existence of this sequence would imply that p_2 was not an accumulation point different from p_1.

In addition, in case p_1 satisfies (6) we can choose x^{n_k} satisfying (6), whereas in case p_1 does not satisfy (6) we can choose x^{n_k} not satisfying (6). Moreover, without loss of generality, we can choose x^{n_k}, for $k\geqslant 1$, verifying one of the following conditions depending on the cases:

(a) $x^{n_k}\in \mathrm{int}(O^\varepsilon_\beta)$, if $p_1\in\mathrm{int}(O^\varepsilon_\beta)$ for some $\beta\in\mathscr{P}(M)$ or if $x^q\in\mathrm{int}(O^\varepsilon_\beta)$, $\forall q>N$, for some $N>0$ and $\beta\in\Sigma(p_1)$.
(b) $x^{n_k}\in O^\varepsilon_\beta\cap O^\varepsilon_{\beta'}$, if $x^q\in O^\varepsilon_\beta\cap O^\varepsilon_{\beta'}$, $\forall q>N$, for some $N>0$ and $\beta,\beta'\in\Sigma(p_1)$.
(c) $x^{n_k}\in O^\varepsilon_\beta\cap O^\varepsilon_{\beta'}$ and $x^{n_k+1}\in\mathrm{int}(O^\varepsilon_\beta)$ ($x^{n_k+1}\in\mathrm{int}(O^\varepsilon_{\beta'})$), if $x^q\in\mathrm{int}(O^\varepsilon_\beta)$ exists $\forall N>0$ and $q>N$ and $x^q\notin\mathrm{int}(O^\varepsilon_\beta)$ $\forall q>N$ for some $\beta,\beta'\in\mathscr{P}(M)$.
(d) $x^{n_k}\in O^\varepsilon_\beta\cap O^\varepsilon_{\beta'}$ and $x^{n_k+1}\in O^\varepsilon_{\beta''}\cap O^\varepsilon_{\beta'''}$, with $\beta,\beta',\beta'',\beta'''\in\mathscr{P}(M)$, otherwise. (Observe that this case considers the sequence $\{x^q\}_{q\in\mathbb{N}}$ such that $x^q\in O^\varepsilon_{\beta_q}\cap O^\varepsilon_{\beta'_q}$, with $\beta_q,\beta'_q\in\mathscr{P}(M)$, $\forall q>N$, and $N>0$. Hence, since the number of bisector lines is finite there exist $\beta,\beta',\beta'',\beta'''\in\mathscr{P}(M)$ (not necessarily all pairwise different) such that we can select a subsequence verifying this condition.)

Notice that this choice of the subsequence $\{x^{n_k}\}$ implies, when x^{n_k} satisfies (6), that $x^{n_k+1}=\Phi_\beta(x^{n_k})$ for the cases (a) and (c) and $x^{n_k+1}=\Psi^{\alpha^*,s}(x^{n_k})$ for the case (b), for all $k\geqslant 1$. Analogously, we can provide an expression of x^{n_k+1} when x^{n_k} does not satisfy (6).

Moreover, the function $T_\varepsilon(\cdot)$ is continuous. For the cases (a) and (c), the continuity follows from the definition of $\phi_\beta(\cdot)$, and in the case (b), by Lemma 4.5. In the case (d), x^{n_k+1} is given by the intersection of $[x^{n_k},\hat{\Psi}^{\alpha^*,s}(x^{n_k})]$ and the bisector line given by $O^\varepsilon_{\beta''}\cap O^\varepsilon_{\beta'''}$. Hence, since $\hat{\Psi}^{\alpha^*,s}(x^{n_k})$ is a continuous function and the bisector line is fixed, the function of x^{n_k} that defines this intersection point is also continuous.

We prove the convergence only for $x^{n_k+1}=\Psi^{\alpha^*,s}(x^{n_k})$, for all $k\geqslant 1$. The remaining cases can be proven by using an argument similar to this one.

Since $\Psi^{\alpha^*,s}(\cdot)$ is continuous, we get that

$$\lim_{k\to\infty} T_\varepsilon(x^{n_k})=\lim_{k\to\infty}\Psi^{\alpha^*,s}(x^{n_k})=\Psi^{\alpha^*,s}(p_1)=T_\varepsilon(p_1).$$

On the other hand, by hypothesis, $\{x^{n_k+1}\}_{k\geqslant 1}=\{\Psi^{\alpha^*,s}(x^{n_k})\}_{k\geqslant 1}$ does not belong to B_1. Thus, $\Psi^{\alpha^*,s}(p_1)$ cannot belong to the interior of B_1. Therefore, $p_1\neq\Psi^{\alpha^*,s}(p_1)$, then, by Lemma 4.6, $F(p_1)>F(\Psi^{\alpha^*,s}(p_1))$.

However, since we have assumed that the sequence does not converge in a finite number of steps, we have that $x^{n_k}\neq\Psi^{\alpha^*,s}(x^{n_k})$ and $\Psi^{\alpha^*,s}(x^{n_k})\neq x^{n_{k+1}}$, then applying Lemma 4.6 we get

$$F_\varepsilon(x^{n_k})>F_\varepsilon(T_\varepsilon(x^{n_k}))=F_\varepsilon(\Psi^{\alpha^*,s}(x^{n_k}))=F_\varepsilon(x^{n_k+1})>F_\varepsilon(x^{n_{k+1}}).$$

Taking limit when k goes to infinity

$$F_\varepsilon(p_1)\geqslant F_\varepsilon(\Psi^{\alpha^*,s}(p_1))\geqslant F_\varepsilon(p_1), \tag{22}$$

what contradicts that $F_\varepsilon(p_1)>F_\varepsilon(\Psi^{\alpha^*,s}(p_1))$.

Therefore, the sequence contains a unique accumulation point, that is, $\{x^q\}_{q\in\mathbb{N}}$ converges to a unique point. Let x^* be the limit.

Now, we prove that x^* is the optimal solution of Problem (2). By (22), it follows that $F_\varepsilon(p_1)=F_\varepsilon(\Psi^{\alpha^*,s}(p_1))$. Then, by Lemma 4.6, we have that $p_1=\Psi^{\alpha^*,s}(p_1)=T_\varepsilon(p_1)$. By Lemma 4.7 we have that x^* coincides with the optimal solution of Problem (2) and the result follows. □

Notice that in the previous proof we have assumed that the set of points belonging to more than one bisector line is finite (the case where two bisector lines coincide can be considered as only one bisector), that is, the intersection of any two bisector lines is given by isolated points (it can be obtained by an adequate perturbation). Therefore, the subsequence considered in the proof can be chosen in such a way that it does not contain points belonging to more than two bisector lines.

6. Computational results

In this section, we present the computational study for solving Problem (2). The algorithm described in Section 5 was coded in MATHEMATICA and run on a PENTIUM IV computer, with a 1.60 GHz processor and 540 MB RAM.

For each M, we solve 15 different randomly generated problems: demand points with values in $[0,10\,000]$ the weights ω_i and λ_i in $[0,1]$, for $i=1,\dots,M$, with $\lambda_1\leqslant\cdots\leqslant\lambda_M$.

All computational tests were performed for values of the parameter $p=1$, 1.5 and 2. The constant ε used in the hyperbolic approximation is fixed to 0.0001. The stopping rule was $\|x^q-x^{q+1}\|_p\leqslant 10^{-5}$.

Table 1 summarizes the results obtained for the approximated ordered median algorithm for ℓ_p distances, with $p=1.5$. The size of each test problem is indicated in the first column of this table. The next four columns report the Mean, Min, Max and the Standard Deviation of the time needed to solve each test problem in CPU seconds. The remaining four columns give the results for the number

Table 1
Numerical results for $p=1.5$.

M	Time				Iterations			
	Mean	Min	Max	STD.D	Mean	Min	Max	STD.D
100	0.85	0.45	1.56	0.31	18.87	10	25	4.72
500	5.95	2.66	16.48	4.2	18.87	10	28	6.13
1000	23.51	4.75	64.42	17.88	26.93	8	60	16.1
2000	35.19	15.87	67.87	18.88	17.07	8	32	8.07
4000	54.06	27.17	78.17	16.40	13.13	7	20	3.81
8000	126.65	47.33	409.78	109.91	15.27	6	47	12.41
12 000	192.99	81.37	343.00	94.54	15.27	7	27	7.15
16 000	205.47	127.09	439.52	82.03	12.2	8	25	4.41
20 000	380.28	139.01	948.78	232.56	17.54	7	44	10.57
25 000	572.23	199.23	1410.25	222.34	18.41	7	47	12.7
30 000	825.33	529.55	2112.58	254.51	15.15	7	27	10.12

Table 2
Numerical results for $p = 2$.

M	Time				Iterations			
	Mean	Min	Max	STD.D	Mean	Min	Max	STD.D
100	0.79	0.20	1.36	0.40	16.6	7	29	6.28
500	4.69	1.0	12.29	3.05	15.67	7	29	6.4
1000	17.17	5.95	38.91	9.05	19.2	7	40	8.91
2000	37.58	13.39	65.86	18.31	19.6	7	52	12.28
4000	58.06	23.61	87.79	18.46	13.8	6	21	4.18
8000	101.74	64.33	169.02	32.86	11.53	8	20	3.62
12 000	130.87	84.03	258.09	42.84	10.53	7	20	3.20
16 000	257.25	93.62	568.74	156.98	15.0	6	32	8.68
20 000	370.21	130.26	843.97	202.96	19.99	8	42	12.57
25 000	551.35	201.31	1001.25	212.43	17.15	7	47	15.7
30 000	820.83	486.66	2012.74	214.1	16.51	7	37	10.21

Table 3
Numerical results for $p = 1$.

M	Time				Iterations			
	Mean	Min	Max	STD.D	Mean	Min	Max	STD.D
100	0.79	0.19	1.64	0.51	20.67	8	49	11.42
500	5.27	1.69	15.42	4.26	20.87	10	44	9.78
1000	13.77	4.70	32.26	6.99	20.8	9	36	8.95
2000	44.63	11.28	144.45	40.25	23.2	6	75	21.06
4000	60.63	21.95	160.92	40.83	15.47	6	41	10.24
8000	130.70	42.86	310.81	92.37	16.33	6	40	11.27
12 000	226.61	95.81	431.45	98.01	18.33	8	33	7.43
16 000	414.40	109.84	994.97	300.04	23.27	3	62	8.8
20 000	610.2	230.26	1443.07	222.66	24.96	8	35	11.24
25 000	934.85	401.31	1661.78	199.54	20.54	8	40	14.61
30 000	1220.44	486.66	2410.4	248.21	19.11	7	38	12.15

of iterations performed by the method before the stopping rule was satisfied.

Tables 2 and 3 summarize the results obtained for $p = 2$ and 1, respectively.

Notice that the standard deviation of the computational times is relatively bigger for problems with a large number of demand points. This is because the iterates that simultaneously belong to many bisector lines need more time to find a decreasing direction and it is more likely to occur as larger is the number of demand points. In addition, the computational times are lower for the case $p = 2$, because in this case the bisector lines are curves with an easier handling of analytical expressions and the iterations can be computed faster.

For the case $p = 1$, the times are longer because even for the case of the median problem, the rate of convergence is lower when the iterate has a common component with a demand point. Notice that this case was also solved by [4,30] as a particular case of polyhedral norms. They develop a solution procedure based on the iterative resolution of linear programming problems on each linearity domain of the objective function. A detailed analysis of the theoretical computational complexity of the procedure is given in those references. However, the computational times are not reported.

For the sake of completeness, the computational results obtained for solving the k-centrum problem with ℓ_2-norm are included in Table 4. We have considered that the number of components equal to one in the λ-vector is $\lfloor 0.2 \cdot M \rfloor$. Observe that the time and the number of iterations to solve this problem are larger than that needed to solve the standard ordered median problem. We think that this is due to the fact that, in most of the cases analyzed, the sequence obtained by the algorithm is included in a bisector line and, there, the stepsize is lower than in the interior of an ordered region. In addition, in order to solve this problem, we have used the same implementation used to solve the standard convex ordered problem and the results obtained could be improved with a new implementation that exploits the particular properties of k-centrum problem.

Table 4
Numerical results for the k-centrum problem for $p = 2$.

M	Time				Iterations			
	Mean	Min	Max	STD.D	Mean	Min	Max	STD.D
100	1.76	0.62	3.97	1.24	31.73	13	89	24.11
500	5.63	2.04	14.52	3.81	25.73	8	58	13.73
1000	25.32	4.78	96.73	22.35	30.53	6	88	20.32
2000	243.89	53.93	454.27	135.78	123.93	18	229	72.33
4000	326.24	58.03	672.33	205.44	80.2	14	171	51.95
8000	171.00	64.63	380.37	86.66	19.87	8	43	9.24
12 000	275.05	132.03	547.36	135.49	20.47	11	41	9.54
16 000	419.30	132.50	1451.8	331.98	21.67	8	80	18.12
20 000	581.27	146.67	1201.78	292.65	21.99	8	42	12.72
25 000	855.54	259.51	1951.22	212.33	22.52	7	46	15.24
30 000	1231.23	686.88	2512.76	214.39	18.31	7	39	11.54

Table 5
Numerical results for the k-centrum problem for $p = 1.5$.

M	Time				Iterations			
	Mean	Min	Max	STD.D	Mean	Min	Max	STD.D
100	2.24	0.60	10.34	2.77	35.4	8	152	38.84
500	7.78	2.15	16.47	4.07	33.47	13	68	15.78
1000	24.86	3.29	67.22	17.11	33.53	8	62	15.69
2000	245.99	51.93	470.65	138.98	103.35	20	209	62.39
4000	341.43	57.03	662.55	207.54	90.12	15	170	50.95
8000	182.30	59.63	399.61	96.95	29.98	7	45	10.81
12 000	275.15	112.03	555.64	155.92	22.23	10	45	10.36
16 000	445.65	120.12	1399.81	301.01	25.56	7	78	17.01
20 000	591.77	150.91	1403.72	295.96	24.44	8	41	12.11
25 000	851.55	247.73	1995.12	202.25	27.53	8	43	10.41
30 000	1355.38	625.31	2223.12	158.45	19.51	7	41	12.94

Table 6
Numerical results for the k-centrum problem for $p = 1$.

M	Time				Iterations			
	Mean	Min	Max	STD.D	Mean	Min	Max	STD.D
100	1.71	0.46	6.12	1.75	31.13	10	131	30.21
500	13.97	2.27	48.25	13.77	41.6	11	121	28.47
1000	45.95	3.35	69.42	19.33	35.25	9	59	14.03
2000	355.67	53.84	495.34	155.02	115.29	20	201	60.24
4000	445.98	59.88	631.28	237.62	95.22	14	175	55.53
8000	295.04	58.55	421.10	99.59	32.04	7	65	15.81
12 000	479.59	153.31	554.69	96.76	25.94	10	55	10.01
16 000	649.67	220.52	1451.55	352.25	29.64	7	79	19.39
20 000	798.88	350.63	1703.45	345.34	22.56	7	46	10.10
25 000	957.23	447.48	2392.34	265.23	26.94	8	53	12.37
30 000	1559.43	725.25	2824.22	225.96	21.34	7	49	13.29

Tables 5 and 6 summarize the results obtained for solving the k-centrum problem with $p = 1.5$ and 1, respectively.

Finally, we have also compared the solution times of our procedure for the special case of the Weber problem $\lambda_1 = \cdots = \lambda_M = 1$ with the classical approximated Weiszfeld algorithm (see [20]). The extra computational effort induced by (then superfluous) the decomposition into the ordered regions is considerable (depending on the values of p and the sizes of the problem, the solution times can vary from one to 20 times more than the classical method). Moreover, this procedure need approximately half of the number of iterations used by the classical method.

7. Concluding remarks

In this paper, we have presented a resolution procedure for the convex ordered median problem with ℓ_p-norms. This approach becomes a very robust methodology because of the flexibility of this objective function, which allows us to deal simultaneously with many classical models of Location Theory as well as with new ones. Indeed, this paper provides a common methodology to solve median, center and cent-dian problems, among others. Moreover, this procedure allows us to solve well-known types of problems for which currently no resolution method has been published, for example, the k-centrum problem with ℓ_p distances.

On the other hand, in the literature, solution procedures for the ordered median problem have only been developed for the case of polyhedral gauges. Therefore, this paper is the first approach to deal with ordered median problems using smooth gauges. Hence, this paper opens a new avenue of research to deal with ordered location problems using these types of gauges.

Finally, after reading this paper, one might suggest that the original problem could be solved with the application of a Diagonal algorithm by selecting a decreasing sequence $\{\varepsilon n\}_{n\in\mathbb{N}}$. However, the sequence generated by this method could be non-convergent. For instance, it would be possible to find a sequence, such that, $x^q \notin B_p^{\varepsilon q}(a_i, a_j)$, $x^{q+1} \in B_p^{\varepsilon q}(a_i, a_j)$ and $x^{q+1} \notin B_p^{\varepsilon_{q+1}}(a_i, a_j)$ for $q > N$, with $N > 0$. In this case, the sequence does not leave the approximated bisector $B^{\varepsilon q}(a_i, a_j)$ and consequently it might not converge.

Appendix A.

Lemma A.1. *For any $\alpha \in [0,1]$ we have that $\Psi^{\alpha,s}(x^q) \in O^{\varepsilon}_{\beta(s)} \cup O^{\varepsilon}_{\beta'(s)}$.*

Proof. First, we prove that $\|\mathscr{D}^s_\alpha(x^q)\|_p \leqslant \|\eta\|_p$, for any $\alpha \in [0,1]$. Define $g_j(\alpha) = \mathscr{D}^s_{\alpha,j}(x^q)$ (see (12)), for $j = 1, 2$, then

$$g_j(\alpha) = \frac{(1-\alpha)\nabla_j F_{\varepsilon,\beta(s)}(x^q) + \alpha\nabla_j F_{\varepsilon,\beta'(s)}(x^q)}{(1-\alpha)C_{\beta(s)j}(x^q) + \alpha C_{\beta'(s)j}(x^q)}.$$

Hence $g_j(\alpha)$ is a linear rational function of α and does not have vertical asymptote because $C_{\beta(s)j} > 0$ and $C_{\beta'(s)j} > 0$, for $j = 1, 2$. Therefore, $|g_j(\alpha)| \leqslant \max\{|g_j(0)|, |g_j(1)|\}$, for $j = 1, 2$, and $\alpha \in [0,1]$. Moreover, for $\alpha = 0$, $G^{\alpha,s}_\varepsilon(x) = F_{\varepsilon,\beta(s)}(x)$, then $|\mathscr{D}^s_{0,j}(x)| = |D^{\beta(s)}_j| \leqslant \eta_j(x)$ and for $\alpha = 1$, $G^{\alpha,s}_\varepsilon(x) = F_{\varepsilon,\beta'(s)}(x)$, then $|\mathscr{D}^s_{1,j}(x)| = |D^{\beta'(s)}_j| \leqslant \eta_j(x)$. Hence,

$$|\mathscr{D}^s_{\alpha,j}(x^q)| \leqslant \max\{|\mathscr{D}^s_{0,j}(x^q)|, |\mathscr{D}^s_{1,j}(x^q)|\} = \eta_j(x),$$
$$j = 1, 2, \quad \forall \alpha \in [0,1].$$

Therefore,

$$\|\Psi^{\alpha,s}(x^q) - x^q\|_p \leqslant \frac{\omega_{\beta_{t+1}}\|h_\varepsilon(x - a_{\beta_{t+1}})\|_p - \omega_{\beta_t}\|h_\varepsilon(x - a_{\beta_t})\|_p}{\omega_{\beta_{t+1}} + \omega_{\beta_t}} - 2^{1/p}\varepsilon^{1/2} \leqslant d_p(x^q, B^\varepsilon_p(a_{\beta_t}, a_{\beta_{t+1}})),$$

for all $t \neq s$, $t \in \{1, \ldots, M-1\}$ and the result follows. □

Lemma A.2. *For $x^q \in B^\varepsilon_p(a_{\beta_s}, a_{\beta_{s+1}})$ and $x^q \notin B^\varepsilon_p(a_{\beta_t}, a_{\beta_{t+1}})$ $\forall t \neq s$, such that $\Phi^s_{\beta(s)}(x^q) \notin O^\varepsilon_{\beta(s)}$ and $\Phi^s_{\beta'(s)}(x^q) \notin O^\varepsilon_{\beta'(s)}$, with $\beta(s) = (\beta_1, \ldots, \beta_M)$ and $\beta'(s) = (\beta_1, \ldots, \beta_{s-1}, \beta_{s+1}, \beta_s, \beta_{s+2}, \ldots, \beta_M)$, there exists at least one value $\alpha^* \in (0,1)$ such that $\Psi^{\alpha^*,s}(x^q) \in B^\varepsilon_p(a_{\beta_s}, a_{\beta_{s+1}}) \cap (O^\varepsilon_{\beta(s)} \cup O^\varepsilon_{\beta'(s)})$.*

Proof. Given x^q, the function $\xi_{x^q}(\alpha)$ (see (13)) is continuous. In addition, if $\alpha = 0$, we have that $\Psi^{\alpha,s}(x^q) = \Phi_{\beta(s)}(x^q) \in O^\varepsilon_{\beta'(s)}$, and then $\xi_{x^q}(0) > 0$. On the other hand, if $\alpha = 1$ we have that $\Psi^{\alpha,s}(x^q) = \Phi_{\beta'(s)}(x^q) \in O^\varepsilon_{\beta(s)}$ and then $\xi_{x^q}(1) < 0$. Hence, using the continuity of $\xi_{x^q}(\alpha)$, there exists a value $\alpha^* \in (0,1)$ such that $\xi_{x^q}(\alpha^*) = 0$, that is,

$$\omega_{\beta_s}\|h_\varepsilon(\Psi^{\alpha^*,s}(x^q) - a_{\beta_s})\|_p = \omega_{\beta_{s+1}}\|h_\varepsilon(\Psi^{\alpha^*,s}(x^q) - a_{\beta_{s+1}})\|_p.$$

Therefore, by Lemma A.1 we have that $\Psi^{\alpha^*,s}(x^q) \in B^\varepsilon_p(a_{\beta_s}, a_{\beta_{s+1}}) \cap (O^\varepsilon_{\beta(s)} \cup O^\varepsilon_{\beta'(s)})$, and the result follows. □

Lemma A.3. *The sequence generated by the algorithm given in* (21) *is bounded.*

Proof. Let $\{x^q\}_{q\in\mathbb{N}}$ be the sequence generated by the algorithm given in (21). Notice that either x^q is obtained by a modified gradient method with stepsize lower than the one of the Weiszfeld algorithm or x^q belongs to a sufficiently small neighborhood of x^{q-1}. In any case, since the Weiszfeld algorithm gives a sequence contained in the convex hull of the existing facilities, we have that $\{x^q\}_{q\in\mathbb{N}}$ is bounded. □

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