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A NOTE ON L^2-SUMMAND VECTORS IN DUAL SPACES

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Abstract. It is shown that every L^2-summand vector of a dual real Banach space is a norm-attaining functional. As consequences, the L^2-summand vectors of a dual real Banach space can be determined by the L^2-summand vectors of its predual; for every $n \in \mathbb{N}$, every real Banach space can be equivalently renormed so that the set of norm-attaining functionals is n-lineable; and it is easy to find equivalent norms on non-reflexive dual real Banach spaces that are not dual norms.

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1. Introduction and background. A vector e of a real Banach space X is said to be an L^2-summand vector if there exists a closed vector subspace M of X such that $X = \mathbb{R}e \oplus_2 M$; in other words, $\|\lambda e + m\|^2 = \|\lambda e\|^2 + \|m\|^2$ for every $\lambda \in \mathbb{R}$ and every $m \in M$. If $e \neq 0$, then the functional $e^* \in X^*$ such that $e^*(e) = 1$ and $M = \ker(e^*)$ is called the L^2-summand functional associated to e. It satisfies $\|e^*\| = \frac{1}{\|e\|}$, where e^* is an L^2-summand vector of X^* and $X^* = \mathbb{R}e^* \oplus_2 \ker(\widehat{e})$, where $\widehat{e}$ denotes the element e in the bidual X^{**} (note that the L^2-summand functional associated to e^* is $\widehat{e}$.) We refer the reader to [**1**] and [**2**] for a wider perspective about L^2-summand vectors.

In this paper, it is shown that if e^* is an L^2-summand vector of the dual Banach space X^*, then e^* must be a norm-attaining functional. From this fact, we conclude several consequences such as the following.

(1) The L^2-summand vectors of a dual real Banach space can be determined by the L^2-summand vectors of its predual.
(2) For every $n \in \mathbb{N}$, every real Banach space can be equivalently renormed so that the set of norm-attaining functionals is n-lineable.
(3) It is easy to find equivalent norms on non-reflexive dual real Banach spaces that are not dual norms.

2. Main result and consequences.

THEOREM 2.1. *Let X be a real Banach space and consider an L^2-summand vector $e^* \in \mathsf{S}_{X^*}$. Then, there exists an L^2-summand vector $e \in \mathsf{S}_X$ such that $e^*(e) = 1$.*

Proof. Let us denote $X^* = \mathbb{R}e^* \oplus_2 \ker(e^{**})$, where $e^{**} \in \mathsf{S}_{X^{**}}$ is the L^2-summand functional associated to e^*. By Goldstine's theorem, for every $n \in \mathbb{N}$, there exists $x_n \in X$

so that $\|\widehat{x_n}\| \leq 1$ and

$$1 - e^*(x_n) = |e^{**}(e^*) - \widehat{x_n}(e^*)| \leq \frac{1}{n}.$$

Now, $\widehat{x_n} = e^*(x_n)e^{**} + (\widehat{x_n} - e^*(x_n)e^{**})$; therefore

$$\begin{aligned} 1 &\geq e^*(x_n)^2 + \|\widehat{x_n} - e^*(x_n)e^{**}\|^2 = e^*(x_n)^2 \\ &\quad + \sup\{(\widehat{x_n} - e^*(x_n)e^{**})(\lambda e^* + m^*) : \lambda \in \mathbb{R}, m^* \in \ker(e^{**}), \lambda^2 + \|m^*\|^2 \leq 1\}^2 \\ &= e^*(x_n)^2 + \sup\{m^*(x_n) : m^* \in \ker(e^{**}), \|m^*\|^2 \leq 1\}^2, \end{aligned}$$

and hence,

$$\begin{aligned} \frac{2}{n} &\geq (1 - e^*(x_n))(1 + e^*(x_n)) \\ &= 1 - e^*(x_n)^2 \\ &\geq \sup\{m^*(x_n) : m^* \in \ker(e^{**}), \|m^*\|^2 \leq 1\}^2. \end{aligned}$$

Now, let us see that the sequence $(\widehat{x_n})_{n\in\mathbb{N}}$ converges to e^{**}, which will conclude the proof, since in that case $e^{**} \in \widehat{X}$ and e^* is norm-attaining. For every $n \in \mathbb{N}$, we have

$$\begin{aligned} \|e^{**} - \widehat{x_n}\| &= \sup\{(e^{**} - \widehat{x_n})(\lambda e^* + m^*) : \lambda \in \mathbb{R}, m^* \in \ker(e^{**}), \lambda^2 + \|m^*\|^2 \leq 1\} \\ &= \sup\{\lambda(1 - e^*(x_n)) - m^*(x_n) : \lambda \in \mathbb{R}, m^* \in \ker(e^{**}), \lambda^2 + \|m^*\|^2 \leq 1\} \\ &\leq \sup\{1 - e^*(x_n) - m^*(x_n) : m^* \in \ker(e^{**}), \|m^*\|^2 \leq 1\} \\ &\leq \frac{1}{n} + \sqrt{\frac{2}{n}}. \end{aligned}$$

As a consequence, $(\widehat{x_n})_{n\in\mathbb{N}}$ converges to e^{**} and the proof is completed. □

REMARK 2.2. In [**1**], it is proved that the set L^2_X of all L^2-summand vectors of a real Banach space X is a closed vector subspace (in fact, it is a Hilbert subspace), that is, L^2-complemented in X (that is, there exists a closed vector subspace M of X such that $X = \mathsf{L}^2_X \oplus_2 M$). In addition, it is shown that $M = \bigcap\{\ker(e^*) : e \in \mathsf{L}^2_X\}$, where each e^* is the L^2-summand functional associated to each e.

REMARK 2.3. Recall that given a smooth Banach space X, the dual map of X is the map $\mathsf{J} : X \longrightarrow X^*$ such that, for every $x \in X$, $\mathsf{J}(x)$ is the unique element in X^* such that $\|\mathsf{J}(x)\| = \|x\|$ and $\mathsf{J}(x)(x) = \|x\|^2$. The book [**4**] is an excellent reference for dual maps in smooth spaces.

COROLLARY 2.4. *Let X be a real Banach space. Then,*

(1) *the map*

$$\begin{aligned} \mathsf{L}^2_X &\longrightarrow \mathsf{L}^2_{X^*} \\ e &\longmapsto e^*\|e\|^2, \end{aligned} \tag{2.1}$$

where e^* denotes the L^2-summand functional associated to e, is a surjective linear isometry and

(2) $\mathsf{L}^2_{X^{**}} = \mathsf{L}^2_{\widehat{X}}$.

Proof.

(1) Let $\mathsf{J} : \mathsf{L}^2_X \longrightarrow (\mathsf{L}^2_X)^*$ denote the dual map. Since L^2_X is a Hilbert space, we have that J is a surjective linear isometry. Now, given any $\mathsf{J}(e) \in (\mathsf{L}^2_X)^*$, let $\phi(\mathsf{J}(e))$ denote a unique element of X^* such that $\phi(\mathsf{J}(e))|_{\mathsf{L}^2_X} = \mathsf{J}(e)$ and $\phi(\mathsf{J}(e))|_M = 0$, where $X = \mathsf{L}^2_X \oplus_2 M$. Consider the map $\phi : (\mathsf{L}^2_X)^* \longrightarrow X^*$. It is easy to check that ϕ is a linear isometry. Let us show that the image of ϕ is $\mathsf{L}^2_{X^*}$. In the first place, take any $e \in \mathsf{L}^2_X$. We will show that $\phi(\mathsf{J}(e)) = e^*\|e\|^2$. Since $e^*\|e\|^2|_M = 0$, it will be sufficient to show that $\mathsf{J}(e) = \phi(\mathsf{J}(e))|_{\mathsf{L}^2_X} = e^*\|e\|^2_{\mathsf{L}^2_X}$. We have that $\|e^*\|e\|^2\| = \|e\|$ and $e^*\|e\|^2(e) = \|e\|^2$; therefore, $e^*\|e\|^2|_{\mathsf{L}^2_X} = \mathsf{J}(e)$, and hence, $e^*\|e\|^2 = \phi(\mathsf{J}(e))$. In the second place, take any $e^* \in \mathsf{L}^2_{X^*}$ with norm 1. According to Theorem 2.1, there exists $e \in \mathsf{L}^2_X$ of norm 1 such that $e^*(e) = 1$. Similarly as above, $e^*|_{\mathsf{L}^2_X} = \mathsf{J}(e)$, and hence, $e^* = \phi(\mathsf{J}(e))$. Finally, the map (2.1) is exactly $\phi \circ \mathsf{J}$, and thus, it is a surjective linear isometry.

(2) Trivially, we have that $\mathsf{L}^2_{\widehat{X}} \subseteq \mathsf{L}^2_{X^{**}}$. If $e^{**} \in \mathsf{L}^2_{X^{**}}$ and $\|e^{**}\| = 1$, then by Theorem 2.1, there is $e^* \in \mathsf{L}^2_{X^*}$ with $\|e^*\| = 1$ such that $e^{**}(e^*) = 1$. By applying the same argument, we deduce the existence of $e \in \mathsf{L}^2_X$ with $\|e\| = 1$ such that $e^*(e) = 1$. Finally, $e^{**} = \widehat{e}$. □

REMARK 2.5. Recall that a subset M of a Banach space is said to be n-lineable, where $n \in \mathbb{N}$, if $M \cup \{0\}$ contains a vector subspace of dimension n. We refer the reader to [**3**] for a wider perspective of lineability.

COROLLARY 2.6. *Let X be a real Banach space. For every $n \in \mathbb{N}$, X can be equivalently renormed so that the set of norm-attaining functionals of X^* is n-lineable.*

Proof. Let us fix $n \in \mathbb{N}$ and denote by $\mathsf{NA}(X)$ the set of norm-attaining functionals on X. According to [**2**], X can be equivalently renormed so that L^2_X is n-lineable. Since L^2_X and $\mathsf{L}^2_{X^*}$ are linearly isometric by Corollary 2.4, we deduce that $\mathsf{L}^2_{X^*}$ is n-lineable under this equivalent norm. Finally, Theorem 2.1 assures that $\mathsf{L}^2_{X^*} \subseteq \mathsf{NA}(X)$, and thus, $\mathsf{NA}(X)$ is n-lineable as well. □

REMARK 2.7. Recall that given any normable real topological vector space X, an equivalent norm $|\!|\!| \cdot |\!|\!|$ on its dual X^* is a dual norm (that is, it comes from a norm on X) if and only if Goldstine's theorem holds, in other words, the set $\{\widehat{x} \in X^{**} : |\!|\!|\widehat{x}|\!|\!|^* \leq 1\}$ is ω^*-dense in $\{x^{**} \in X^{**} : |\!|\!|x^{**}|\!|\!|^* \leq 1\}$. We refer the reader to [**5**] for a wider perspective.

COROLLARY 2.8. *Let X be a non-reflexive real Banach space X. Let $e^* \in \mathsf{S}_{X^*}$ be such that there exists $e^{**} \in \mathsf{S}_{X^{**}} \backslash \mathsf{S}_{\widehat{X}}$ with $e^{**}(e^*) = 1$. Then, the equivalent norm on X^* given by*

$$|\!|\!| x^* |\!|\!| = \sqrt{e^{**}(x^*)^2 + \|x^* - e^{**}(x^*)\, e^{**}\|^2}$$

for all $x^* \in X^*$, is not a dual norm on X^*.

Proof. Otherwise, assume that $|\!|\!| \cdot |\!|\!|$ is a dual norm. Then, there exists an equivalent norm $|\cdot|$ on X such that $|\cdot|^* = |\!|\!| \cdot |\!|\!|$. Now, e^* is an L^2-summand vector of norm 1 of $(X^*, |\!|\!| \cdot |\!|\!|)$; therefore, by Theorem 2.1, there exists $e \in (X, |\cdot|)$ with $|e| = 1$ such that $e^*(e) = 1$. Finally, both e^{**} and $\widehat{e}$ are the L^2-summand functionals associated to e^*, and thus, $e^{**} = e$, which is impossible. □

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