



Optimal actions in problems with convex loss functions

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ABSTRACT

Researches in Bayesian sensitivity analysis and robustness have mainly dealt with the computation of the range of some quantities of interest when the prior distribution varies in some class. Recently, researchers' attention turned to the loss function, mostly to the changes in posterior expected loss and optimal actions. In particular, the search for optimal actions under classes of priors and/or loss functions has lead, as a first approximation, to consider the set of nondominated actions. However, this set is often too big to take it as the solution of the decision problem and some criteria are needed to choose an optimal alternative within the nondominated set. Some authors recommended to choose the conditional Γ -minimax or the posterior regret Γ -minimax alternative within the set of all possible alternatives. These criteria are quite controversial since they could lead to actions with huge relative increase in posterior expected loss with respect to Bayes actions. To overcome such drawback, we propose a new method, based on the smallest relative error, to choose the least sensitive action and to discriminate alternatives within the nondominated set when the decision maker is interested in diminishing the relative error. We study how to compute the least sensitive action when we consider classes of convex loss functions. Furthermore, we obtain its relation with other proposed solutions: nondominated, minimax and posterior regret minimax actions. We conclude the paper with an example on the estimation of the mean of a Poisson distribution.

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1. Introduction

Sensitivity analysis is an important part in the application of any mathematical model to real problems. In many fields it is worth studying how changes in the input parameters affect the output from a model. A large, but not exhaustive, number of examples can be found in [23]. Similarly, the sensitivity analysis is essential in Bayesian analysis and decision theory. The early 1990s was the golden age of Bayesian sensitivity analysis, in that many statisticians were highly active in research in the area, and rapid progress was being achieved. See [17] for a thorough review of those accomplishments.

We consider the standard Bayesian decision theoretic framework for statistical problems. Let X be an observation from a distribution P_θ with density $p_\theta(x)$, where θ is in the parameter space Θ . We consider priors π in a class of distributions Γ and loss functions L in a class \mathcal{L} . The actions a are considered in the action space \mathcal{A} .

Let π_x denote the posterior density when x is observed, $m_\pi(x)$ the (prior) marginal density and $\rho(\pi, L, a)$ the posterior expected loss of a , i.e.

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$$\rho(\pi, L, a) = \frac{\int_{\Theta} L(a, \theta) p_{\theta}(x) \pi(\theta) d\theta}{m_{\pi}(x)} = E^{\pi_x}[L(a, \theta)].$$

Definition 1. For any $(L, \pi) \in \mathcal{L} \times \Gamma$, a Bayes action corresponding to (L, π) , denoted by $a_{(L,\pi)}^*$, is an action that minimizes $\rho(\pi, L, a)$ in \mathcal{A} , i.e.

$$\rho(\pi, L, a_{(L,\pi)}^*) = \inf_{a \in \mathcal{A}} \rho(\pi, L, a).$$

Since the conclusions of the analysis depend on π and L , their choice in $\mathcal{L} \times \Gamma$ must be very careful and the consequences must be somehow measured. Excellent surveys of sensitivity analyses with respect to the prior are [3,4,13], whereas sensitivity with respect to the loss function has been considered in [7,9,11], among others. The key idea in these, and related, works is the choice of a class of priors and/or loss functions instead of unique prior and loss function. Sensitivity is then analyzed considering some measures, mainly the range spanned by a quantity of interest as the prior distribution (and/or the loss function) varies in the assigned class. The main differences among works in Bayesian sensitivity are about the choice of the classes and the sensitivity measures.

The most common measures of interest are the posterior mean, the posterior variance and the posterior expected loss, see e.g. [5,14,15,21,24,25], among others. The relevance of the calibration of these measures is stressed in many studies but, unfortunately, there are only few proposals to deal with it and find “objective” tools to interpret their values. See [19] for one of them.

Martín et al. [11] show that the use of measures based on the range can provide misleading conclusions in the sensitivity analysis with respect to the loss function. Different approaches have been proposed, e.g. selecting the “best” (with respect to some criterion) optimal alternative. The choice of the set of Bayes alternatives can be unsuitable since there are Bayes alternatives that provide very large posterior expected losses when the choice of the prior distribution or the loss function is not the correct one. In [18], Ríos Insua and Criado show that the nondominated actions are the optimal solutions of the decision problem when considering a class of loss functions. Since this set is usually very large, it is very important to choose an alternative in it. Some authors, e.g. Betrò and Ruggeri [6] and Ríos Insua [16], studied the conditional and posterior regret Γ -minimax approaches which could be used as criteria to choose an action in a class, e.g. the set of nondominated actions. As shown in the paper, these actions could lead to a huge relative increase in posterior expected loss with respect to Bayes actions. Therefore, in this paper we propose and study a sensitivity measure, extension of one introduced by Ruggeri and Sivaganesan [22], that overcomes such drawback. The measure leads to an optimality criterion and optimal actions, called least sensitive ones, which are compared with Bayes and nondominated ones. We provide also results useful in implementing algorithms for the actual computation of least sensitive actions.

The structure of the paper is the following. In Section 2 we motivate with an example the need for a new sensitivity measure and then we introduce our measure, comparing it with others existing in the literature. From this measure we define the least sensitive alternative (LS). We dedicate Section 3 to the relation between LS and other alternatives: Bayes, nondominated, ... Section 4 provides results to characterize and calculate LS actions under classes of convex loss functions. We illustrate the previous results with an example about estimation of the parameter of a Poisson distribution when the prior distribution belongs to different parametric classes. The paper ends with some concluding remarks.

2. A new sensitivity measure

In this section we review available methods of discrimination among alternatives and propose a new one, that we consider more effective in choosing “optimal” actions and decision rules when the relative increase in posterior expected loss or Bayes risk, respectively, is the main concern for the decision maker.

2.1. The least sensitive action

Betrò and Ruggeri [6] and Vidakovic [26] consider the conditional Γ -minimax alternatives, that can be easily extended to $\mathcal{L} \times \Gamma$ -minimax:

Definition 2. a^* is a conditional $\mathcal{L} \times \Gamma$ -minimax action if

$$\sup_{(L,\pi) \in \mathcal{L} \times \Gamma} \rho(\pi, L, a^*) = \inf_{a \in \mathcal{A}} \sup_{(L,\pi) \in \mathcal{L} \times \Gamma} \rho(\pi, L, a).$$

Thus, the conditional $\mathcal{L} \times \Gamma$ -minimax principle is a conservative criterion, in the sense that it protects against the worst possible cases.

Ríos Insua in [16] and Dey and Micheas in [8] propose another alternative, the former paper for classes of priors and the latter one for classes of loss functions. The extension of the criterion to joint classes of priors and losses is straightforward. Let $r(\pi, L, a)$ be the posterior regret of an alternative a , defined as

$$r(\pi, L, a) = \rho(\pi, L, a) - \rho(\pi, L, a_{(L,\pi)}^*).$$

Definition 3. a_M is the posterior regret $\mathcal{L} \times \Gamma$ -minimax (PRLGM) action if

$$\sup_{(L,\pi) \in \mathcal{L} \times \Gamma} r(\pi, L, a_M) = \inf_{a \in \mathcal{A}} \sup_{(L,\pi) \in \mathcal{L} \times \Gamma} r(\pi, L, a).$$

As a conservative criterion, an action is taken such that it protects against the worst possible discrepancy between the corresponding posterior expected losses and the optimal ones, as priors and losses vary in the class $\mathcal{L} \times \Gamma$.

However, the choice of these alternatives often produces much greater posterior expected loss than the Bayes alternatives. The relative error can be very large, as shown in the following example.

Example 1. Let $\Gamma = \{\pi_1, \pi_2, \pi_3\}$ be a class of three prior distributions such that their posterior distributions are $\pi_{1x} \sim N(0, 10)$, $\pi_{2x} \sim N(1, 0.1)$ and $\pi_{3x} \sim N(0.5, 5)$. We suppose that the preferences are modelled by the quadratic loss function L (the Bayes actions are the posterior means). The posterior expected losses for the alternatives a can be described by parabolas centered on the posterior means, i.e.

$$\begin{aligned} \rho(\pi_1, a) &= (a - 0)^2 + 10, \\ \rho(\pi_2, a) &= (a - 1)^2 + 0.1, \\ \rho(\pi_3, a) &= (a - 0.5)^2 + 5. \end{aligned}$$

The conditional Γ -minimax action is $a^* = 0$ (Bayes action for π_1), whereas the posterior regret Γ -minimax action is $a_M = 0.5$ (the middle point of the minimum and the supremum of the Bayes actions, Bayes action for π_3), see Fig. 1. However, it is worth mentioning that $\rho(\pi_2, L, a^*) = 1.1$ while $\rho(\pi_2, L, a_{(L,\pi_2)}^*) = 0.1$, leading to a 1000% increase in the posterior expected loss due to the Bayes alternative. Similarly, $\rho(\pi_2, L, a_M) = 0.350$, denoting a 250% increase. We will see later that, in this example, there exist alternatives with much smaller relative error.

As shown by the previous example, it is important that any proposed criterion controls the relative increase in posterior expected loss with respect to the one from Bayes alternatives.

First, we define a new sensitivity measure, which extends the measure proposed by Ruggeri and Sivaganesan [22] who considered a quadratic loss function rather than a general one, like here.

Definition 4. Given a pair $(L, \pi) \in \mathcal{L} \times \Gamma$ and the action a , we define the sensitivity of a respect to the pair (L, π) , which will be denoted by $S(\pi, L, a)$, as

$$S(\pi, L, a) = \frac{\rho(\pi, L, a) - \rho(\pi, L, a_{(L,\pi)}^*)}{\rho(\pi, L, a_{(L,\pi)}^*)}.$$

Therefore, we consider the relative increase in posterior expected loss when we have an action a instead of the Bayes action. Note that this measure is scale invariant. If we consider the quadratic loss function:

$$S(\pi, L, a) = \frac{(a - \mu_\pi)^2}{V^\pi},$$

being μ_π and V^π the posterior mean and variance of π .

From now onwards, we assume $\rho(\pi, L, a_{(L,\pi)}^*) > 0$ for all $(L, \pi) \in \mathcal{L} \times \Gamma$, and \mathcal{A} will be a bounded closed interval. Regarding notations, we will use $S(\pi, a)$ or $S(L, a)$ when the loss function or the prior are, respectively, known.

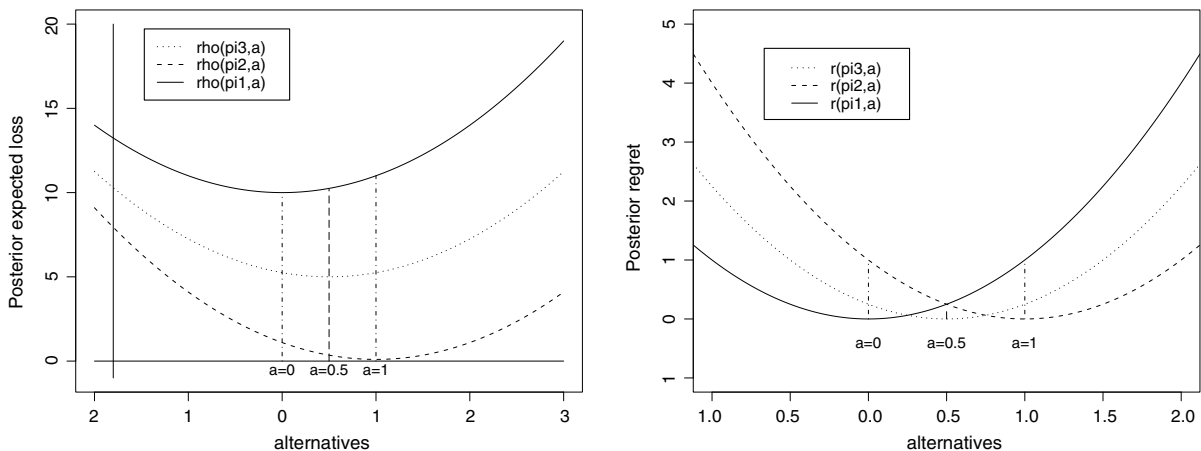


Fig. 1. Posterior expected loss and regret for each distribution.

Finally, we propose a criterion which chooses the alternative minimizing the relative error:

Definition 5. a_s is the least sensitive alternative (LS) for $\mathcal{L} \times \Gamma$ if

$$S(a_s) = \sup_{(L,\pi) \in \mathcal{L} \times \Gamma} S(\pi, L, a_s) = \inf_{a \in \mathcal{A}} \sup_{(L,\pi) \in \mathcal{L} \times \Gamma} S(\pi, L, a),$$

where $S(a)$ denotes the sensitivity of an action a with respect to $\mathcal{L} \times \Gamma$.

In the next example we see how our LS alternative can be better than conditional Γ -minimax and posterior regret Γ -minimax actions, when we are interested in minimizing the relative error with respect to Bayes action.

Example 2 (Continuation of Example 1). It is easy to prove that the sensitivity for each action a with respect to the class Γ is

$$S(a) = \begin{cases} \frac{(a-1)^2}{0.1}, & \text{if } a \leq 10/11; \\ \frac{(a-0)^2}{10}, & \text{otherwise.} \end{cases}$$

If we restrict the set of alternatives \mathcal{A} to the three Bayes actions $\{0, 0.5, 1\}$, then $a_{\pi_2}^* = 1$ is the least sensitive action, see Fig. 2. In fact, we have $S(a_{\pi_1}^*) = 10$, $S(a_{\pi_3}^*) = 2.5$ and $S(a_{\pi_2}^*) = 0.1$, denoting, respectively, 1000%, 250% and 10% increases with respect to the optimal expected loss (the posterior expected loss of the Bayes action).

If we consider the set of alternatives $\mathcal{A} = \mathbb{R}$, then the least sensitive alternative is $a_s = \frac{10}{11}$, with sensitivity $S(a_s) = \frac{10}{11^2} = 0.0826$, i.e. “only” a maximum increase of 8.26%. In this case the LS action is not a Bayes alternative for the priors in Γ .

In general, the LS actions are not Bayes alternatives for any pair $(L, \pi) \in \mathcal{L} \times \Gamma$. The same situation occurs for other Γ -minimax criteria and we refer to Vidakovic [26, and the references therein], for a justification of the Γ -minimax approach.

2.2. The least sensitive decision rule

In this section, we define the LS decision rule and we show its utility comparing it with the Γ -minimax and the Γ -minimax regret decision rules in an example. We assume here that the loss function L is known, whereas the prior π varies in a class Γ . These rules consider the Bayes risk $r(\pi, \delta)$ of a decision rule δ with respect to a prior π ,

$$r(\pi, \delta) = \int_{\{x: m_\pi(x) > 0\}} \rho(\pi, \delta(x)) dF^m(x),$$

where $F^m(x)$ is the marginal distribution of X . A (nonrandomized) decision rule is any function of the sample space into \mathcal{A} , while a decision rule δ^* is said to be Γ -minimax if

$$\sup_{\pi} r(\pi, \delta^*) = \inf_{\delta} \sup_{\pi} r(\pi, \delta).$$

A decision rule $\hat{\delta}$ is said to be Γ -minimax regret rule if

$$\sup_{\pi} [r(\pi, \hat{\delta}) - r(\pi)] = \inf_{\delta} \sup_{\pi} [r(\pi, \delta) - r(\pi)],$$

where $r(\pi)$ is the Bayes risk for π , i.e.:

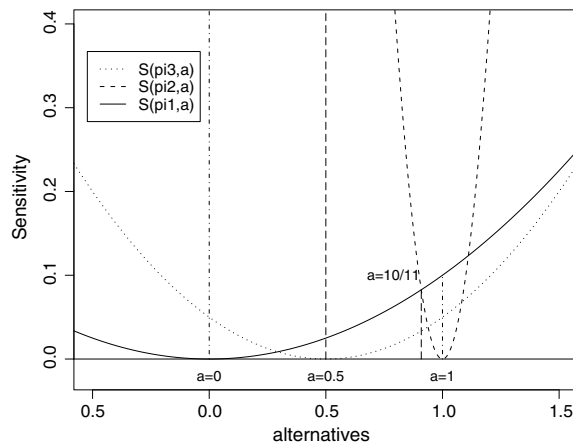


Fig. 2. Sensitivity of a with respect to each distribution.

$$r(\pi) = r(\pi, \delta_\pi) = \inf_{\delta} r(\pi, \delta)$$

and δ_π is the Bayes rule.

Similarly, we can define the least sensitive decision rule δ_s .

Definition 6. A rule δ_s is said to be the least sensitive decision rule for Γ if

$$\sup_{\pi} Sr(\pi, \delta_s) = \inf_{\delta} \sup_{\pi} Sr(\pi, \delta)$$

with

$$Sr(\pi, \delta) = \int_{\{x:m_\pi(x)>0\}} S(\pi, \delta(x)) dF^m(x).$$

Randomized decision rules are not considered since Theorem 3 in [3], ensures that only nonrandomized rules should be considered when \mathcal{A} is convex and $L(a, \theta)$ is a convex loss function of a .

The next example clarifies the concepts.

Example 3 [6]. Let X be a Bernoulli random variable with density

$$p_\theta(x) = \theta^x(1 - \theta)^{1-x},$$

where $x = 0, 1$ and the unknown parameter $\theta \in [0, 1]$. We consider the quadratic loss function and two prior distributions $\pi_1 \sim \mathcal{U}(0, 1)$ and π_2 with density

$$\pi_2(\theta) = \begin{cases} 3/2, & \text{if } 0 \leq \theta \leq 1/2; \\ 1/2, & \text{if } 1/2 < \theta \leq 1. \end{cases}$$

Given $x = 1$, then it follows that, for all $a \in \mathcal{A}$,

$$\rho(\pi_1, a) = \left(a - \frac{2}{3}\right)^2 + \frac{1}{18} \quad \text{and} \quad \rho(\pi_2, a) = \left(a - \frac{5}{9}\right)^2 + \frac{43}{648}.$$

Thus, the Bayes actions are $2/3$ and $5/9$ for π_1 and π_2 , respectively, i.e. the posterior means of π_1 and π_2 . Therefore, the minimum expected losses coincide with the posterior variances, $1/18$ for π_1 and $43/648$ for π_2 , respectively.

It is easy to prove that the sensitivity for all $a \in \mathcal{A}$ is

$$S(a) = \begin{cases} 18(a - 2/3)^2, & \text{if } a \leq a_s; \\ 648/43(a - 5/9)^2, & \text{if } a \geq a_s, \end{cases}$$

where $a_s = (10\sqrt{43} + 86)/(18\sqrt{43} + 129)$ is the LS actions. We can see that the conditional Γ -minimax action is $a^* = 9/16$ and the posterior regret Γ -minimax action is $a_M = 11/18$.

When $x = 0$ is given, then,

$$\rho(\pi_1, a) = \left(a - \frac{1}{3}\right)^2 + \frac{1}{18} \quad \text{and} \quad \rho(\pi_2, a) = \left(a - \frac{4}{15}\right)^2 + \frac{67}{1800}.$$

The sensitivity for all $a \in \mathcal{A}$ is

$$S(a) = \begin{cases} 18(a - 1/3)^2, & \text{if } a \leq a_s; \\ 1800/67(a - 4/15)^2, & \text{if } a \geq a_s, \end{cases}$$

where, in this case, $a_s = (\sqrt{67} + 8)/(3\sqrt{67} + 30)$. The conditional Γ -minimax action is $a^* = 1/3$ and the posterior regret Γ -minimax action is $a_M = 3/10$. The Bayes actions are $1/3$ and $4/15$ for π_1 and π_2 , respectively.

As shown in [6], the Γ -minimax decision rule is δ^* , with $\delta^*(1) = 2/3$ and $\delta^*(0) = 1/3$, and the Γ -minimax regret rule is $\hat{\delta}$ such that $\hat{\delta}(1) = 0.617326$ and $\hat{\delta}(0) = 0.29527$. In this example, the LS actions never coincide with the Γ -minimax and the Γ -minimax regret rules. The next table shows the posterior expected loss and the sensitivity corresponding to a_s , a^* , a_M , and the actions determined by the rules δ^* , $\hat{\delta}$ and δ_s , respectively:

Optimal ($X = 0$)	Value	$\rho(\pi_1, \cdot)$	$\rho(\pi_2, \cdot)$	Sensitivity
Bayes for π_1	0.3333	0.0556	0.0417	0.1194
Bayes for π_2	0.2667	0.0600	0.0372	0.0799
Conditional Γ -minimax (a^*)	0.3333	0.0556	0.0417	0.1194
Γ -minimax regret (a_M)	0.3	0.0567	0.0383	0.0299
LS action (a_s)	0.2966	0.0569	0.0381	0.0243

Optimal ($X = 1$)	Value	$\rho(\pi_1, \cdot)$	$\rho(\pi_2, \cdot)$	Sensitivity
Bayes for π_1	0.6666	0.0556	0.0787	0.1858
Bayes for π_2	0.5556	0.0679	0.0664	0.2220
Conditional Γ -minimax (a^*)	0.5625	0.0664	0.0664	0.1953
Γ -minimax regret (a_M)	0.6111	0.0586	0.0694	0.0556
LS action (a_S)	0.6136	0.0584	0.0697	0.0507

Optimal	Value($X = 0$)	Value($X = 1$)	$Sr(\pi, \delta)$
Γ -minimax rule (δ^*)	0.3333	0.6666	0.1444
Γ -minimax regret rule ($\hat{\delta}$)	0.2952	0.6174	0.0353
LS rule (δ_S)	0.2896	0.6229	0.0345

It is worth mentioning that the Γ -minimax regret leads to actions whose sensitivity is very close to the one of the LS actions, where conditional Γ -minimax leads to actions with high sensitivity. It is logical that it happens this since Γ -minimax does not consider the value of the other prior distributions and loss functions, but only those in which the maximum is minimized.

3. LS actions and nondominated set

In this section we study the relation between LS actions and nondominated sets, which have been considered in Bayesian robustness by Martín et al. [11] and Martín and Arias [10], among others. This aspect was not considered in [22] where the interest was mostly on asymptotic properties of the sensitivity measure.

Ríos Insua and Criado [18] give foundations for robust Bayesian analysis, considering a preference relation \preceq over \mathcal{A} , the set of alternatives. We consider the following preference relation on the set $\mathcal{L} \times \Gamma$:

Given a, b , two alternatives in the set of alternatives \mathcal{A} , then $b \preceq a$ if and only if

$$\rho(\pi, L, a) \leq \rho(\pi, L, b)$$

for all $(L, \pi) \in \mathcal{L} \times \Gamma$.

Definition 7. Let $a, b \in \mathcal{A}$ be such that $a \neq b$; we will say that a dominates b if for all $(L, \pi) \in \mathcal{L} \times \Gamma$ it holds that $\rho(\pi, L, a) \leq \rho(\pi, L, b)$, and for some $(L_0, \pi_0) \in \mathcal{L} \times \Gamma$ it holds the strict inequality, $\rho(\pi_0, L_0, a) < \rho(\pi_0, L_0, b)$.

Note that a dominates b if and only if $a \prec b$ (that is, $a \preceq b$ and $\neg(b \preceq a)$). Therefore, an alternative $a \in \mathcal{A}$ is nondominated if there is no other alternative $b \in \mathcal{A}$ such that b dominates a .

Proposition 1. For any dominated action $a \in \mathcal{A}$ there is another action $b \in \mathcal{A}$ such that $S(a) \geq S(b)$.

Proof. If a is dominated, let $b \in \mathcal{A}$ such that it dominates a . Then, for all $(L, \pi) \in \mathcal{L} \times \Gamma$, it follows that

$$\rho(\pi, L, b) \leq \rho(\pi, L, a)$$

with strict inequality for some pair $(L_0, \pi_0) \in \mathcal{L} \times \Gamma$. Then

$$S(\pi, L, b) \leq S(\pi, L, a)$$

for all $(L, \pi) \in \mathcal{L} \times \Gamma$ and therefore $S(b) \leq S(a)$. \square

Corollary 1. If the LS action exists and is unique then it is nondominated.

With a similar proof we can see that the conditional Γ -minimax and the Γ -minimax regret actions are nondominated alternatives too.

Example 4 (Continuation of Example 3). Example 3 presented a LS action which was not Bayes. Furthermore, applying results in [2], it follows that the set of nondominated alternatives is $[5/9, 2/3]$ when $x = 1$, whereas it becomes $[4/15, 1/3]$ when $x = 0$. In this case the set of Bayes actions is strictly contained in the nondominated one.

In general there are not inclusion relations between the Bayes actions and the nondominated actions sets. It is easy to see that, if there is a unique Bayes alternative a for all $(L, \pi) \in \mathcal{L} \times \Gamma$, then a is the unique nondominated alternative. Although this is the best case, it seldom occurs, see [1] for more results.

We now prove that, under some general conditions, the LS actions are Bayes actions. The first step will be the search for the nondominated set.

3.1. Nondominated set under convex loss functions

In this Section we suppose that the set of alternatives is $\mathcal{A} = \mathbb{R}$. The extension of the results to intervals of \mathbb{R} is straightforward.

Let \mathcal{L} be a class of convex loss functions in $a \in \mathcal{A}$, such that for all $(L, \pi) \in \mathcal{L} \times \Gamma$, the set of Bayes alternatives $B_{(L,\pi)}$ is non empty. Then, it is easy to prove that, for all $(L, \pi) \in \mathcal{L} \times \Gamma$, $\rho(\pi, L, a)$ and $S(\pi, L, a)$ are convex too. Moreover, $\rho(\cdot, L, \pi)$ is strictly decreasing in $(-\infty, a_{(L,\pi)})$, constant in $[a_{(L,\pi)}, a^{(L,\pi)}]$ and strictly increasing in $(a^{(L,\pi)}, +\infty)$, where

$$a_{(L,\pi)} = \inf_{a \in B_{(L,\pi)}} a,$$

$$a^{(L,\pi)} = \sup_{a \in B_{(L,\pi)}} a.$$

It is well known (see, e.g. [20]) that, if a function is convex in a set, then it is continuous in its interior. Therefore, the alternatives $a_{(L,\pi)}$ and $a^{(L,\pi)}$ are Bayes alternatives.

It can be easily shown that the set of nondominated alternatives is included in the interval $[\mu_*, \mu^*]$, where μ_* and μ^* are the infimum and supremum of the set of Bayes alternatives, respectively, i.e.,

$$\mu_* = \inf_{(L,\pi) \in \mathcal{L} \times \Gamma} a_{(L,\pi)},$$

$$\mu^* = \sup_{(L,\pi) \in \mathcal{L} \times \Gamma} a^{(L,\pi)}.$$

The width of the interval $[\mu_*, \mu^*]$, often called “range”, is the most common sensitivity measure; see [13] and the references therein. Moreover, such interval coincides with $\mathcal{N}\mathcal{D}(\mathcal{A})$, the nondominated set in \mathcal{A} under strictly convex loss functions. This is not true when using loss functions which are not strictly convex, as proved in the following theorem in [2].

Theorem 1. *Let \mathcal{L} be a class of convex loss functions in \mathcal{A} and Γ a class of probability distributions such that for all $(L, \pi) \in \mathcal{L} \times \Gamma$, the set of Bayes alternatives $B_{(L,\pi)}$ is non empty. Let $a_* = \inf_{(L,\pi) \in \mathcal{L} \times \Gamma} a_{(L,\pi)}$ and let $a^* = \sup_{(L,\pi) \in \mathcal{L} \times \Gamma} a^{(L,\pi)}$. Then, if a_* is smaller than a^* , it holds that*

$$(a_*, a^*) \subseteq \mathcal{N}\mathcal{D}(\mathcal{A}) \subseteq [a_*, a^*].$$

Otherwise $\mathcal{N}\mathcal{D}(\mathcal{A}) = [a^*, a_*]$.

3.2. LS and the Bayes actions under quadratic loss functions

We consider the quadratic loss function $L(a, \theta) = (a - \theta)^2$. As we have seen before, it is easy to see that

$$S(\pi, L, a) = \frac{(a - \mu_\pi)^2}{V^\pi},$$

being μ_π and V^π the posterior mean and variance of π . In this case $S(\pi, L, a)$ coincides with the relative sensitivity given by Ruggeri and Sivaganesan [22], using $h(\theta) = \theta$.

We will now see when the LS actions are Bayes actions with respect to the prior.

Proposition 2. *Let $\mathcal{L} = \{L_k : L_k(a, \theta) = k(a - \theta)^2, k > 0\}$ be the class of quadratic loss functions and let Γ be a class of prior distributions contained in $\{\pi : -\infty < \int_{\Theta} \theta d\pi_x(\theta) = \mu_\pi < \infty\}$ and let μ_* and μ^* be the values*

$$\mu_* = \inf_{\pi \in \Gamma} \mu_\pi,$$

$$\mu^* = \sup_{\pi \in \Gamma} \mu_\pi,$$

then the nondominated set is $[\mu_*, \mu^*]$.

Proof. See [2]. \square

Proposition 3. *Under the same conditions as Proposition 2 and the convexity of the class of prior distributions, the set of Bayes actions $\mathcal{B}(\mathcal{A})$ is such that*

$$(\mu_*, \mu^*) \subseteq \mathcal{B}(\mathcal{A}) \subseteq \mathcal{N}\mathcal{D}(\mathcal{A}) = [\mu_*, \mu^*].$$

Proof. We start proving that the class of posterior distributions is convex too. Given $\pi_1, \pi_2 \in \Gamma$, for any $\lambda \in [0, 1]$, we consider the distribution $\pi = \lambda\pi_1 + (1 - \lambda)\pi_2 \in \Gamma$. It holds, see [5],

$$m_\pi(x) = \lambda m_{\pi_1}(x) + (1 - \lambda)m_{\pi_2}(x)$$

and

$$\pi_x(\theta) = \lambda(x)\pi_{1x}(\theta) + (1 - \lambda(x))\pi_{2x}(\theta),$$

being

$$m_\pi(x) = \int \pi(\theta)f(x|\theta) d\theta$$

and

$$\lambda(x) = \frac{\lambda m_{\pi_1}(x)}{m_\pi(x)} = \frac{\lambda m_{\pi_1}(x)}{\lambda m_{\pi_1}(x) + (1 - \lambda)m_{\pi_2}(x)} \in [0, 1].$$

Considering $\lambda(x)$ as a function in λ for x fixed, then $\lambda(x)$ is increasing, continue and it maps the interval $[0, 1]$ into $[0, 1]$. Thus, the class of posterior distributions is convex.

On the other hand, if μ_π is the posterior mean of π , we have that

$$\mu_\pi = \lambda(x)\mu_{\pi_1} + (1 - \lambda(x))\mu_{\pi_2}.$$

Moreover, given a value $\mu \in [\mu_{\pi_1}, \mu_{\pi_2}]$, it is easy to prove that there exists $\alpha \in [0, 1]$, such that μ is the posterior mean for the distribution $\pi = \alpha\pi_1 + (1 - \alpha)\pi_2$.

As $\mu \in [\mu_{\pi_1}, \mu_{\pi_2}]$, there exists $\lambda \in [0, 1]$, such that $\mu = \lambda\mu_{\pi_1} + (1 - \lambda)\mu_{\pi_2}$, thus, it is sufficient to take

$$\alpha = \frac{\lambda m_{\pi_2}(x)}{\lambda m_{\pi_2}(x) + (1 - \lambda)m_{\pi_1}(x)}$$

to prove the result. \square

This proposition is very interesting since it shows that any nondominated alternative is a Bayes action for some pair $(L, \pi) \in \mathcal{L} \times \Gamma$, with the possible exception of some extreme points of $\mathcal{N}\mathcal{D}(\mathcal{A})$. Then, the LS action is Bayes action with respect to some pair $(L, \pi) \in \mathcal{L} \times \Gamma$, except perhaps when the LS action is μ_* or μ^* .

4. LS actions under convex loss functions

From now onwards, we will consider a unique convex loss function L in $a \in \mathcal{A}$. Similar results are valid with a class of convex loss functions. We now provide results useful to implement an algorithm to compute LS actions.

Let Π_a denote the set of all densities π_a such that $S(\pi_a, a) = S(a)$. This set can be interpreted as the set of the “relativity least favorable priors” with respect to action a .

Proposition 4. Let $\mathcal{A} = \mathbb{R}$ or a closed and bounded interval of \mathbb{R} . Then, $S(a)$ has at least one minimum a_s in \mathcal{A} . If Π_a is not empty for any $a \in \mathcal{A}$ and the loss function is strictly convex, then there is a unique LS action.

Proof. As we saw, if $L(a, \theta)$ is convex in $a \in \mathcal{A}$, then for all $\pi \in \Pi$, $S(\pi, a)$ is convex, so that $S(a)$ (i.e. the supremum of convex functions) is convex too. If $L(a, \theta)$ is strictly convex, $S(a)$ is strictly convex if the supremum is achieved in Γ , for any $a \in \mathcal{A}$. \square

As a first step to calculate the LS actions, we have the following:

Lemma 1. If at $a_0 \in \mathcal{A}$ there exists $\pi_0 \in \Pi_{a_0}$ such that $a_0 \leq a_{\pi_0}^*$, then

$$S(a_0) \leq S(a) \text{ for all } a < a_0. \quad (1)$$

If at $a_0 \in \mathcal{A}$ there exists $\pi_0 \in \Pi_{a_0}$ such that $a_0 \geq a_{\pi_0}^*$, then

$$S(a_0) \leq S(a) \text{ for all } a > a_0. \quad (2)$$

If the loss function is strictly convex, then the strictly inequality holds in (1) and (2).

Proof. Due to the convexity of $S(\pi, \cdot)$, for all $a < a_0$, it follows $S(\pi_0, a) \geq S(\pi_0, a_0)$, whereas the inequality is strict if L is strictly convex. Then $S(a_0) \leq \sup_{\pi \in \Gamma} S(\pi, a) = S(a)$. \square

The above lemma, based on [6], provides a useful tool for discarding subintervals of \mathcal{A} in the search of the LS actions, even if the loss function is not strictly convex.

From now onwards, $L(a, \theta)$ will be assumed a strictly convex function of a . Lemma 1 and Proposition 4 give immediately:

Proposition 5. If at a_0 there exist π_1 and $\pi_2 \in \Pi_{a_0}$ such that

$$a_{\pi_1}^* \leq a_0 \leq a_{\pi_2}^*,$$

then $a_0 = a_s$. It follows that, at any $a_0 \neq a_s$, either $a_{\pi}^* > a_0$ or $a_{\pi}^* < a_0$ for all $\pi \in \Pi_{a_0}$.

Proof. By Lemma 1, for all $a \neq a_0$, $S(a) \geq S(a_0)$. Then a_0 is LS action. \square

The converse is not necessarily true, but the following result holds:

Proposition 6. a_s is the unique alternative in \mathcal{A} such that

$$a_{\pi}^* > a \text{ for some } \pi \in \Pi_a \quad \forall a < a_s;$$

$$a_{\pi}^* < a \text{ for some } \pi \in \Pi_a \quad \forall a > a_s.$$

Proof. Similar to the proof of Proposition 3 in [6]. \square

Proposition 6 gives a constructive way for obtaining the LS actions. Starting from a given a , then some $\pi \in \Pi_a$ is found, along with the corresponding Bayes action. If the Bayes action is larger (smaller) than a , then the candidates for a_s are to be sought to the right (left) of a . It is clear that in this way it is possible to provide an algorithm for bracketing a_s within any prefixed accuracy.

Example 5. Suppose that X_1, X_2, \dots, X_n is a sample from a uniform distribution $\mathcal{U}(0, \theta)$, the prior $\pi_{\alpha_0, \beta}$ belongs to the following class of Pareto distributions

$$\Gamma_1 = \{ \pi \sim \mathcal{P}(\alpha_0, \beta), \beta \in [\beta_0, \beta_1] \subset \mathbf{R}^+, \alpha_0 > 2 \text{ fixed} \}$$

and the loss function is the quadratic loss function. The Bayes alternatives are the means of the posterior distributions $\mathcal{P}(\alpha, \max(X_{(n)}, \beta))$, where $\alpha = \alpha_0 + n$ and $X_{(n)} = \max_{i=1, \dots, n} X_i$. Thus, the nondominated set is

$$\mathcal{N}\mathcal{D}(\mathcal{A}) = \begin{cases} [\frac{\alpha}{\alpha-1} X_{(n)}, \frac{\alpha}{\alpha-1} \beta_1] & \text{if } \beta_0 < X_{(n)} < \beta_1, \\ [\frac{\alpha}{\alpha-1} \beta_0, \frac{\alpha}{\alpha-1} \beta_1] & \text{if } X_{(n)} < \beta_0, \\ \frac{\alpha}{\alpha-1} X_{(n)} & \text{if } X_{(n)} > \beta_1. \end{cases}$$

If $X_{(n)} > \beta_1$ the LS action is the unique nondominated alternative $\frac{\alpha}{\alpha-1} X_{(n)}$. Otherwise, the nondominated actions are $a_{\beta_a} = \frac{\alpha}{\alpha-1} \beta_a$, where β_a belongs to the interval $[\max(\beta_0, X_{(n)}), \beta_1]$. The sensitivity of each alternative a

$$S(a) = \sup_{\beta \in [\beta_0, \beta_1]} S_{\beta}(a) = \sup_{\beta \in [\beta_0, \beta_1]} \alpha(\alpha - 2) \frac{(\beta_a - \beta)^2}{\beta^2},$$

has a minimum in $\beta = \beta_a$ and the supremum is achieved at β_0 or β_1 .

The next table shows the LS actions for $\alpha_0 = 3$, $\beta_0 = 55$, $\beta_1 = 59$, $\epsilon = 10^{-5}$ and for several samples:

n	$X_{(n)}$	$\mathcal{N}\mathcal{D}(\mathcal{A})$	LS actions	Sensitivity
10	57.702	[62.511, 63.917]	63.206	0.018
100	58.915	[59.493, 59.578]	59.536	0.005
1000	59.928	59.988	59.988	0
10000	59.996	60.002	60.002	0

5. Numerical example: estimating a poisson mean

In this example, suggested by Męczarski and Zieliński in [12], the LS action is found analytically in some interesting situations and we compare the LS action with the conditional Γ -minimax and posterior regret Γ -minimax alternatives.

Suppose that X_1, X_2, \dots, X_n is a sample from a Poisson distribution $\mathcal{P}(\lambda)$ and the prior $\pi_{\alpha, \beta}$ belongs to one of the following classes of Gamma distributions

$$\Gamma_1 = \{ \pi \sim \mathcal{G}(\alpha, \beta), \alpha \in [\alpha_1, \alpha_2] \subset \mathbf{R}^+, \beta > 0 \text{ fixed} \},$$

$$\Gamma_2 = \{ \pi \sim \mathcal{G}(\alpha, \beta), \beta \in [\beta_1, \beta_2] \subset \mathbf{R}^+, \alpha > 0 \text{ fixed} \},$$

$$\Gamma_3 = \{ \pi \sim \mathcal{G}(\alpha, \beta), \alpha \in [\alpha_1, \alpha_2] \subset \mathbf{R}^+, \beta \in [\beta_1, \beta_2] \subset \mathbf{R}^+ \},$$

where α and β are, respectively, the shape and scale parameters. We are interested in estimating the parameter λ under quadratic loss function. Thus, for any action a and $\pi \sim \mathcal{G}(\alpha, \beta)$, it is easy to see that

$$\rho(\pi, a) = \left(\frac{\alpha + n\bar{X}}{\beta + n} - a \right)^2 + \frac{\alpha + n\bar{X}}{(\beta + n)^2},$$

the posterior regret is

$$r(\pi, a) = \left(\frac{\alpha + n\bar{X}}{\beta + n} - a \right)^2$$

and $a_\pi = \frac{\alpha + n\bar{X}}{\beta + n}$. The sensitivity of a with respect to $\pi \sim \Gamma(\alpha, \beta)$ is then

$$S(\pi, a) = \frac{(\alpha + n\bar{X} - a(\beta + n))^2}{\alpha + n\bar{X}}.$$

By Proposition 2, we have the following result.

Corollary 2. *If the class of prior distributions is Γ_3 ,*

$$\mathcal{N}\mathcal{D}(\mathcal{A}) = \left[\frac{\alpha_1 + n\bar{X}}{\beta_2 + n}, \frac{\alpha_2 + n\bar{X}}{\beta_1 + n} \right].$$

In this case the nondominated set and the set of Bayes actions coincide. Thus, the LS actions for these models are Bayes actions for some prior distribution π . The reason is that the parameter space is convex and Bayes actions are continuous functions of the parameters.

The following results are obtained using simple algebra.

Corollary 3. *If the class of priors is Γ_3 , then the sensitivity of $a \in \mathcal{A}$ is*

$$S(a) = \begin{cases} \frac{\rho(\pi_{\alpha_2, \beta_1}, a)}{\rho(\pi_{\alpha_2, \beta_1}, a_{\pi_{\alpha_2, \beta_1}})} - 1, & \text{if } a \leq a_s; \\ \frac{\rho(\pi_{\alpha_1, \beta_2}, a)}{\rho(\pi_{\alpha_1, \beta_2}, a_{\pi_{\alpha_1, \beta_2}})} - 1, & \text{if } a \geq a_s, \end{cases}$$

being

$$a_s = \frac{(\alpha_2 + n\bar{X})\sqrt{\alpha_1 + n\bar{X}} + (\alpha_1 + n\bar{X})\sqrt{\alpha_2 + n\bar{X}}}{(\beta_1 + n)\sqrt{\alpha_1 + n\bar{X}} + (\beta_2 + n)\sqrt{\alpha_2 + n\bar{X}}},$$

the LS action, which is the Bayes action under the prior

$$\mathcal{G}(\lambda_n(\bar{X})\alpha_2 + (1 - \lambda_n(\bar{X}))\alpha_1, \lambda_n(\bar{X})\beta_1 + (1 - \lambda_n(\bar{X}))\beta_2) \in \Gamma_3,$$

being

$$\lambda_n(\bar{X}) = \frac{\sqrt{\alpha_1 + n\bar{X}}}{\sqrt{\alpha_1 + n\bar{X}} + \sqrt{\alpha_2 + n\bar{X}}}.$$

The sensitivity of a_s is then

$$S(a_s) = \left(\frac{(\beta_2 + n)(\alpha_2 + n\bar{X}) - (\alpha_1 + n\bar{X})(\beta_1 + n)}{(\beta_2 + n)\sqrt{\alpha_2 + n\bar{X}} + (\beta_1 + n)\sqrt{\alpha_1 + n\bar{X}}} \right)^2.$$

We can see in [16], that the posterior regret Γ -minimax action is

$$a_M = \frac{\frac{(\alpha_1\beta_1 + \alpha_2\beta_2)/2n + (\alpha_1 + \alpha_2)/2}{(\beta_1 + \beta_2)/2n + 1} + n\bar{X}}{\frac{\beta_1\beta_2/n + (\beta_1 + \beta_2)/2}{(\beta_1 + \beta_2)/2n + 1} + n}$$

which is the Bayes action under the prior

$$\mathcal{G}\left(\frac{\alpha_1\beta_1 + \alpha_2\beta_2}{2n + 1} + \frac{\alpha_1 + \alpha_2}{2}, \frac{\beta_1\beta_2/n + (\beta_1 + \beta_2)/2}{(\beta_1 + \beta_2)/2n + 1}\right) \in \Gamma_3.$$

Example 6. Suppose that X_1, X_2, \dots, X_n is a sample from a Poisson distribution $\mathcal{P}(\lambda)$ and the prior $\pi_{\alpha, \beta}$ belongs to the following class of Gamma distributions

$$\Gamma_3 = \{\pi \sim \mathcal{G}(\alpha, \beta), \alpha \in [1, 4], \beta \in [2, 3]\}.$$

The nondominated set is, in this case, the closed interval [15.462, 17] and the posterior expected losses, the posterior regret and the sensitivity of the Bayes, posterior regret Γ -minimax and the LS alternatives are shown in the next table for $n = 10$ and $\bar{X} = 20$:

Optimal action	Value	$\rho(\pi_{\alpha_1, \beta_2}, a)$	$\rho(\pi_{\alpha_2, \beta_1}, a)$	Sup. Post. Regret	Sensitivity
$a^*_{\pi_{\alpha_1, \beta_2}}$	15.462	1.189	3.784	2.367	1.671
$a^*_{\pi_{\alpha_2, \beta_1}}$	17	3.556	1.417	2.367	1.990
a_M	16.231	1.781	2.008	0.592	0.498
a_s	16.197	1.730	2.061	0.645	0.455

Remark 1. If the class of prior distributions is Γ_1 , then the sensitivity of $a \in \mathcal{A}$ is

$$S(a) = \begin{cases} \frac{\rho(\pi_{\alpha_2}, a)}{\rho(\pi_{\alpha_2}, a^*_{\pi_{\alpha_2}})} - 1, & \text{if } a \leq a_s; \\ \frac{\rho(\pi_{\alpha_1}, a)}{\rho(\pi_{\alpha_1}, a^*_{\pi_{\alpha_1}})} - 1, & \text{if } a \geq a_s, \end{cases}$$

being $a_s = \frac{\sqrt{(\alpha_1 + n\bar{X})(\alpha_2 + n\bar{X})}}{\beta + n}$ the LS action. The sensitivity the a_s is then

$$S(a_s) = \left(\sqrt{\alpha_2 + n\bar{X}} - \sqrt{\alpha_1 + n\bar{X}} \right)^2 = 2 \left(\frac{(\alpha_2 + n\bar{X}) + (\alpha_1 + n\bar{X})}{2} - \sqrt{(\alpha_1 + n\bar{X})(\alpha_2 + n\bar{X})} \right).$$

Note that, in this case, the LS action is the geometrical mean of the extremes of the nondominated set. However, this result is not always true, as we have seen in [Example 1](#).

We can see too that the sensitivity of a_s does not depend on the parameter β . This is obvious, since $a_s \in \mathcal{N}(\mathcal{D}(\mathcal{A}))$, so there exists $\alpha_s = \sqrt{(\alpha_1 + n\bar{X})(\alpha_2 + n\bar{X})} - n\bar{X} \in [\alpha_1, \alpha_2]$ (which depends on the sample), such that $a_s = \frac{\alpha_s + n\bar{X}}{\beta + n}$. As shown in [\[16\]](#), the posterior regret Γ -minimax is the middle point of the set of nondominated alternatives, that is

$$a_M = \frac{\frac{\alpha_1 + \alpha_2}{2} + n\bar{X}}{\beta + n},$$

which is the Bayes actions under the prior $\mathcal{G}(\frac{\alpha_1 + \alpha_2}{2}, \beta)$. Since $a_M > a_s$, the sensitivity of the (PRGM) is

$$S(a_M) = \frac{\rho(\pi_{\alpha_1}, a_M)}{\rho(\pi_{\alpha_1}, a^*_{\pi_{\alpha_1}})} - 1 = \frac{(\alpha_1 - \alpha_2)^2}{4(\alpha_1 + n\bar{X})}.$$

Remark 2. If the class of prior distributions is Γ_2 , then the sensitivity of $a \in \mathcal{A}$ is

$$S(a) = \begin{cases} \frac{\rho(\pi_{\beta_1}, a)}{\rho(\pi_{\beta_1}, a^*_{\pi_{\beta_1}})} - 1, & \text{if } a \leq a_s; \\ \frac{\rho(\pi_{\beta_2}, a)}{\rho(\pi_{\beta_2}, a^*_{\pi_{\beta_2}})} - 1, & \text{if } a \geq a_s, \end{cases}$$

being $a_s = \frac{\alpha + n\bar{X}}{\frac{\beta_1 + \beta_2}{2} + n}$ the LS action, which is the Bayes action under the prior $\mathcal{G}(\alpha, \frac{\beta_1 + \beta_2}{2})$. The sensitivity of a_s is then

$$S(a_s) = \frac{(\alpha + n\bar{X})(\beta_1 - \beta_2)^2}{4(\frac{\beta_1 + \beta_2}{2} + n)^2}.$$

In this case the posterior regret Γ -minimax is

$$a_M = \frac{\frac{\alpha}{\beta_1 \beta_2} + n\bar{X}}{\frac{(\beta_1 + \beta_2)/n + (\beta_1 + \beta_2)/2}{(\beta_1 + \beta_2)/2 + n} + 1},$$

which is the Bayes action under the prior $\mathcal{G}(\alpha, \frac{\beta_1 \beta_2 / n + (\beta_1 + \beta_2) / 2}{(\beta_1 + \beta_2) / 2n + 1})$. Since $a_M > a_s$, then the sensitivity of the PRGM action is

$$S(a_M) = \frac{\alpha + n\bar{X}}{4} \left(\frac{\beta_2 - \beta_1}{\beta_1 + n} \right)^2.$$

6. Conclusions

We have generalized a sensitivity measure, proposed by Ruggeri and Sivaganesan [\[22\]](#), to address the problem of choosing an action in a set when interested in reducing the relative increase in posterior expected loss with respect to Bayes alter-

natives. We called the resulting actions “least sensitive” and we compared them with other ones, like the conditional Γ -minimax and the posterior regret Γ -minimax actions, showing the shortcomings of the latter ones. The choice of an “optimal” action (with respect to some criterion) is important especially when sensitivity analysis leads to a lack of robustness so that actions must be chosen very carefully. As discussed by some authors, the nondominated actions are the optimal solutions of decision problems under classes of loss functions, whereas Bayes actions are the “classical” solutions in the Bayesian framework. We have therefore compared LS with Bayes and nondominated actions under classes of convex loss function, providing results useful in implementing algorithms for the actual computation of LS actions.

Possible extensions of the current work could lead to the study of LS actions under different classes of loss functions and the study of asymptotic properties of the generalized measure, in the same fashion as in [22]. In a forthcoming paper, we are studying the computation of LS actions under different classes of prior distributions.

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