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Symmetries for a family of Boussinesq equations with nonlinear dispersion

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ABSTRACT

In this paper, we make a full analysis of a family of Boussinesq equations which include nonlinear dispersion by using the classical Lie method of infinitesimals. We consider travelling wave reductions and we present some explicit solutions: solitons and compactons.

For this family, we derive nonclassical and potential symmetries. We prove that the nonclassical method applied to these equations leads to new symmetries, which cannot be obtained by Lie classical method. We write the equations in a conserved form and we obtain a new class of nonlocal symmetries. We also obtain some Type-II hidden symmetries of a Boussinesq equation.

1. Introduction

The Boussinesq equation, which belongs to the KdV family of equations and describes motions of long waves in shallow water under gravity propagating in both directions, is given by

$$u_{tt} = u_{xx} + cu_{xxxx} + (u^2)_{xx} = 0, \tag{1}$$

where $u = u(x,t)$ is a sufficiently often differentiable function, which for $c = -1$ gives the good Boussinesq or well-posed equation, while for $c = 1$ the bad or ill-posed classical equation [4,5].

In [15], Rosenau extended the Boussinesq equation to include nonlinear dispersion to the effect that the new equations support compact and semi-compact solitary structures in higher dimensions,

$$u_{tt} = au_{xx} + (u^{m+1})_{xx} + b[u(u^m)_{xx}]_{xx}, \tag{2}$$

where a and b are arbitrary constants. Eq. (2) describe for $a = 0$ the vibrations of a purely an harmonic lattice and support travelling structures with a compact support [15,16].

One of the most useful point transformations are those which form a continuous group. Lie classical symmetries admitted by nonlinear partial differential equations (PDEs) are useful for finding invariant solutions. In [6], we studied similarity reductions of the generalized Boussinesq equation (2), with a, b, m arbitrary constants and $m \neq 0$. Motivated by the fact that symmetry reductions for many PDEs are known that are not obtained by using the classical Lie method there have been several generalizations of the classical Lie group method for symmetry reductions. Bluman and Cole [2], in their study of symmetry reductions of the linear heat equation, proposed the so-called nonclassical method of group-invariant solutions. In [3],

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Bluman introduced a method to find a new class of symmetries for a PDE when it can be written in a conserved form. These symmetries are nonlocal symmetries which are called *potential* symmetries. In [11], Gandarias introduced a new classes of symmetries for a PDE, which can be written in the form of conservation laws. These symmetries, called *nonclassical potential* symmetries, are realized as nonclassical symmetries of an associated system.

In [10], Clarkson obtained some nonclassical symmetry reductions and exact solutions for a Boussinesq equation. Gandarias and Bruzón [13] applied the Lie group and the nonclassical method to deduce symmetries for another Boussinesq equation.

Two possible methods have been identified in [1] for finding possible PDEs the symmetries of which are inherited in a transformation. In [12], Gandarias proposed to have as differential constraint the side condition from which the reduction has been derived and to derive weak symmetries, that is, Lie classical symmetries of the original equation and the side condition.

The aim of this paper is to make a full analysis of Eq. (2), by using classical symmetries, nonclassical symmetries and nonclassical potential symmetries, and to obtain new solutions. We also obtain some Type-II hidden symmetries of Eq. (2) with $m = 1$ and $a = \lambda^2$.

2. Classical symmetries

To apply the classical method to (2), we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned} \tag{3}$$

where ϵ is the group parameter. Then, we require that this transformation leaves invariant the set of solutions of (2). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{4}$$

We consider the classical Lie group symmetry analysis of Eq. (2). Invariance of Eq. (2) under a Lie group of point transformations with infinitesimal generator (19) leads to the following set of twenty seven determining equations.

The solutions of this system depend on the constants of the equation:

Case 1. If a, b and m are arbitrary constants, the only symmetries admitted by (2) are the group of space and time translations, which are defined by the infinitesimal generators

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}.$$

In the following cases, we obtain extras symmetry, and these symmetry are defined by the following infinitesimal generators:

Case 2. If a is arbitrary and $m = -1$, V_1, V_2 and

$$V_3^1 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

Case 3. If m is arbitrary and $a = 0$, V_1, V_2 and

$$V_3^2 = t\frac{\partial}{\partial t} - \frac{2u}{m}\frac{\partial}{\partial u}.$$

Case 4. If $a = 0$ and $m = -1$, V_1, V_2, V_3^2 and

$$V_4 = x\frac{\partial}{\partial x} - 4u\frac{\partial}{\partial u}.$$

2.1. Optimal system and symmetry reductions

In order to determine solutions of PDE (2) that are not equivalent by the action of the group, we must calculate the one-dimensional optimal system [14]. The generators of the nontrivial one-dimensional optimal system are the set of subalgebras:

Case 2.

$$V_1, \quad V_2, \quad V_1 + V_2, \quad V_3^1.$$

Case 3.

$$V_1, \quad V_2, \quad V_1 + V_2, \quad V_3^2, \quad V_1 + V_3^2.$$

Case 4.

$$V_1, \quad V_2, \quad V_3^2, \quad V_4, \quad V_1 + V_2, \quad V_1 + V_4, \quad V_2 + V_3^3, \quad V_3^3 + V_4.$$

Having determined the optimal system, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi \frac{\partial u}{\partial x} + \tau \frac{\partial u}{\partial t} - \eta = 0. \tag{5}$$

In *case 1*, for $V_1 + V_2$, we obtain travelling wave reductions

$$z = \mu x - \lambda t, \quad u = h(z),$$

where $h(z)$ satisfies

$$\begin{aligned} &bm\mu^4 h^{m+3} h'''' + (4bh^{m+2} h' m^2 - 2bh^{m+2} h' m)\mu^4 h''' + (-h^3 \lambda^2 + (6bh^{m+1}(h')^2 m^3 - 11bh^{m+1}(h')^2 m^2 \\ &\quad + 5bh^{m+1}(h')^2 m)\mu^4 + (h^{m+3} m + h^{m+3} + ah^3)\mu^2)h'' + (bh^m (h')^4 m^4 - 4bh^m (h')^4 m^3 + 5bh^m (h')^4 m^2 \\ &\quad - 2bh^m (h')^4 m)\mu^4 + (3bh^{m+2} m^2 - 2bh^{m+2} m)\mu^4 (h'')^2 + (h^{m+2}(h')^2 m^2 + h^{m+2}(h')^2 m)\mu^2 = 0. \end{aligned} \tag{6}$$

Since Eq. (2) has additional symmetries and the reductions that correspond to V_1 and V_2 have already been derived, we must only determine the similarity variables and similarity solutions corresponding to the remaining generators:

- For V_3^1:

$$z = \frac{x}{t}, \quad u = t^{-2} h(z),$$

where $h(z)$ satisfies the ODE

$$bh^3 h'''' + z^2 h^4 h'' + 6zh^4 h' - 6bh^2 h' h''' - 5bh^2 (h'')^2 + 22bh(h')^2 h'' - ah^4 h'' - 12b(h')^4 + 6h^5 = 0. \tag{7}$$

- For V_3^2:

$$z = x, \quad u = t^{-2/m} h(z),$$

where $h(z)$ satisfies the ODE

$$\begin{aligned} &-bh^m (h')^4 m^6 - h^m (6bh(h')^2 h'' - 4b(h')^4) m^5 - h^m (4bh^2 h' h''' + 3bh^2 (h'')^2 - 11bh(h')^2 h'' + 5b(h')^4 \\ &\quad + h^2 (h')^2) m^4 - h^m (bh^3 h'''' - 2bh^2 h' h''' - 2bh^2 (h'')^2 + (5bh(h')^2 + h^3) h'' - 2b(h')^4 + h^2 (h')^2) m^3 \\ &\quad - h^{m+3} h'' m^2 + 2h^4 m + 4h^4 = 0. \end{aligned}$$

- For $V_1 + V_3^2$:

$$z = x - \ln(t), \quad u = t^{-2/m} h(z),$$

where $h(z)$ satisfies the ODE

$$\begin{aligned} &-bh^m (h')^4 m^6 - 6bh^{m+1} (h')^2 h'' m^5 + 4bh^m (h')^4 m^5 - 4bh^{m+2} h' h''' m^4 - 3bh^{m+2} (h'')^2 m^4 \\ &\quad + 11bh^{m+1} (h')^2 h'' m^4 - 5bh^m (h')^4 m^4 - h^{m+2} (h')^2 m^4 - bh^{m+3} h'''' m^3 + 2bh^{m+2} h' h''' m^3 \\ &\quad + 2bh^{m+2} (h'')^2 m^3 - h^{m+3} h'' m^3 - 5bh^{m+1} (h')^2 h'' m^3 + 2bh^m (h')^4 m^3 - h^{m+2} (h')^2 m^3 \\ &\quad - h^{m+3} h'' m^2 - h^3 h'' m^2 - h^3 h' m^2 - 4h^3 h' m - 2h^4 m - 4h^4 = 0. \end{aligned}$$

- For $V_2 + V_4$:

$$z = xe^{-t}, \quad u = e^{-4t} h(z),$$

where $h(z)$ satisfies the ODE

$$bh^3 h'''' + h^4 h'' z^2 + 9h^4 h' z - 6bh^2 h' h''' - 5bh^2 (h'')^2 + 22bh(h')^2 h'' - 12b(h')^4 + 16h^5 = 0. \tag{8}$$

2.2. Travelling wave solutions

From Eq. (6), we can obtain some exact solutions of Eq. (2) for $a = \left(\frac{\lambda}{\mu}\right)^2$:

- If $b = \frac{1}{\mu^2}$,

$$u(x,t) = \sin^{\frac{1}{m}}(\mu x - \lambda t), \quad u(x,t) = \cos^{\frac{1}{m}}(\mu x - \lambda t). \tag{9}$$

- If $b = -\frac{1}{\mu^2}$,

$$u(x,t) = \sinh^{\frac{1}{m}}(\mu x - \lambda t), \quad u(x,t) = \cosh^{\frac{1}{m}}(\mu x - \lambda t) \tag{10}$$

- If $m = -1$,

$$\begin{aligned} u(x,t) &= \csc(\mu x - \lambda t), \quad u(x,t) = \sec(\mu x - \lambda t), \\ u(x,t) &= \mathrm{csch}(\mu x - \lambda t), \quad u(x,t) = \mathrm{sech}(\mu x - \lambda t) \end{aligned} \tag{11}$$

- If $b = \frac{1}{16\mu^2}$,
  - If $m = -2$,

$$u(x,t) = \csc^2(\mu x - \lambda t), \quad u(x,t) = \sec^2(\mu x - \lambda t). \tag{12}$$

-If $m = -\frac{2}{3}$,

$$u(x,t) = \sin^6(\mu x - \lambda t), \quad u(x,t) = \cos^6(\mu x - \lambda t) \tag{13}$$

- If $b = -\frac{1}{16\mu^2}$,
  -If $m = -2$,

$$u(x,t) = \mathrm{sech}^2(\mu x - \lambda t), \quad u(x,t) = \mathrm{csch}^2(\mu x - \lambda t). \tag{14}$$

-If $m = -\frac{2}{3}$,

$$u(x,t) = \sinh^6(\mu x - \lambda t), \quad u(x,t) = \cosh^6(\mu x - \lambda t). \tag{15}$$

In the following we give some solutions with physical interest:
From (9), with $m = \frac{1}{6}$, and (13) we obtain that

$$u(x,t) = \begin{cases} \sin^6(\gamma(x-t)), & |x-t| \leqslant \frac{\pi}{\gamma}, \\ 0, & |x-t| > \frac{\pi}{\gamma}, \end{cases} \tag{16}$$

is a compacton solution of (2) with $a = 1$, $b = \frac{1}{\gamma^2}$ and $m = \frac{1}{6}$ and for (2) with $a = 1$, $b = \frac{1}{16\gamma^2}$ and $m = -\frac{2}{3}$. In Fig. 1, we plot solution (16) with $\gamma = \frac{1}{4}$ which is a sine-type double compacton solution.

From (10), with $m = -\frac{1}{2}$, and (14) we obtain that

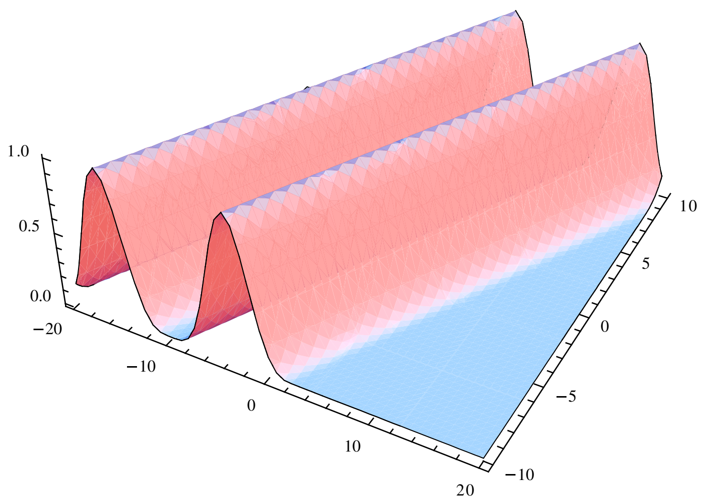

Fig. 1. Solution (16) for $\gamma = \frac{1}{4}$.

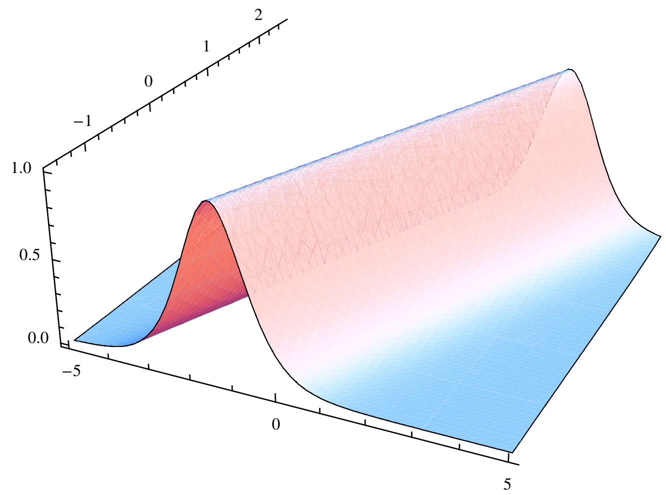

Fig. 2. Solution (17).

$$u(x,t) = \text{sech}^2(x-t), \tag{17}$$

is an exact solution of (2) with $a = 1$, $b = -1$ and $m = -\frac{1}{2}$ and for (2) with $a = 1$, $b = -\frac{1}{16}$ and $m = -2$. In Fig. 2, we plot solution (17) which is a soliton solution.

Solutions ((9), (10), (25), (12)–(15)) are solutions of Eq. (2) and these solutions do not appear in [6].

3. Nonclassical symmetries

The basic idea of the method is that the PDE (2) is augmented with the invariance surface condition

$$\Phi \equiv \xi \frac{\partial u}{\partial x} + \tau \frac{\partial u}{\partial t} - \eta = 0, \tag{18}$$

which is associated to the vector field

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{19}$$

By requiring that both, (2) and (18), are invariant under the transformation with infinitesimal generator (19), an overdetermined nonlinear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$ is obtained. The number of determining equations arising in the nonclassical method is smaller than for the classical method, consequently the set of solutions is, in general, larger than for the classical method. However, the associated vector fields do not form a vector space.

To obtain nonclassical symmetries of (2) we apply the algorithm described in [10] for calculating the determining equations and we use the MACSYMA program symmgrp.max [7]. We can distinguish two different cases:

In the case $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,u) = 1$, and we obtain a set of sixteen determining equations for the infinitesimals $\xi(x,t,u)$ and $\eta(x,t,u)$. Solving this system, we obtain

1. If a, b and m are arbitrary constants

$$\xi = k_1, \quad \eta = 0,$$

where k_1 and k_2 are constants.
2. If m is arbitrary and $a = 0$, $\xi = \frac{k_1}{t+k_2}$, $\eta = -\frac{2u}{m(t+k_2)}$, where k_1 and k_2 are constants.
3. If a is arbitrary and $m = -1$, $\xi = \frac{x+k_1}{t+k_2}$, $\eta = -\frac{2u}{t+k_2}$, where k_1 and k_2 are constants.
4. If $a = 0$ and $m = -1$, $\xi = \frac{-x+k_1k_3}{k_1(t+k_2)}$, $\eta = \frac{2(k_1+2)u}{k_1(t+k_2)}$, where k_1, k_2 and k_3 are constants.

By comparing these symmetries with the symmetries obtained by the classical method given in [6] we can observe that *the nonclassical method applied to* (2) *gives only rise to the classical symmetries*.

In the case $\tau = 0$, without loss of generality, we may set $\xi = 1$ and we obtain one overdetermined system for the infinitesimal η.

The complexity of this system is the reason why we cannot solve it in general. Thus we proceed, by making ansatz on the form of $\eta(x,t,u)$, to solve the system.

For $b = -\frac{1}{m^2}$, with $m = 1, 2$, choosing $\eta = \eta(x, u)$, we find that the infinitesimal generators take the form

$$\xi = 1, \quad \tau = 0, \quad \eta = \frac{u \cosh x}{\sinh x}. \tag{20}$$

It is easy to check that these generators do not satisfy the Lie classical determining equations. Therefore, we obtain the nonclassical symmetry reduction

$$z = t, \quad u = h(t) \sinh x,$$

where $h(t)$ satisfies the following linear second-order ODEs:

- For $m = 1$

$$h'' - ah = 0. \tag{21}$$

- For $m = 2$

$$h'' + 2h^3 - ah = 0. \tag{22}$$

The solutions of Eq. (21) yield the following exact solutions of Eq. (2):

If $a > 0$,

$$u = (k_1 \exp(\sqrt{a}t + k_2 \exp(-\sqrt{a}t)) \sinh x. \tag{23}$$

If $a < 0$,

$$u = (k_1 \cos(\sqrt{-a}\,t) - k_2 \sin(\sqrt{-a}\,t)) \sinh x. \tag{24}$$

If $a = 0$,

$$u = (k_1 t + k_2) \sinh x. \tag{25}$$

After multiplying (22) by $2y'$ and integrating once with respect to z we get

$$(h')^2 = -h^4 + ah^2. \tag{26}$$

This equation is solvable in terms of the Jacobian elliptic functions.

We remark that, when $b = -1$ and $m = 1$ Eq. (2) does not admit any classical symmetry but translations. Consequently, (23)–(25), which *are not* travelling waves reductions, cannot be obtained by Lie classical symmetries.

For $m = 1$, choosing $\eta = \eta(x, t)$, we find the following infinitesimal generators:

$$\xi = 1, \quad \tau = 0, \quad \eta = x\psi_2(t) + \psi_1(t), \tag{27}$$

where $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\frac{d^2\psi_2}{dt^2} - 6\psi_2^2 = 0, \tag{28}$$

$$\frac{d^2\psi_1}{dt^2} - 6\psi_1\psi_2 = 0, \tag{29}$$

respectively. In this case, we obtain the nonclassical symmetry reduction

$$z = t, \quad u = x^2\psi_2(t) + x\psi_1(t) + \psi_0(t),$$

where $\psi_2(t)$ satisfies the Weierstrass elliptic function equation (28) and $\psi_1(t)$ satisfies the Lamé equation (29) [10].

4. Classical potential symmetries

In order to find potential symmetries of (2), we write the equation in a conserved form and the associated auxiliary system is given by

$$\begin{cases} v_x = -u_t, \\ v_t = au_x + (u^{m+1})_x + b[u(u^m)_{xx}]_x. \end{cases} \tag{30}$$

If $(u(x), v(x))$ satisfies (30), then $u(x)$ solves the generalized Boussinesq equation. The basic idea for obtaining *classical potential* symmetries is to require that the infinitesimal generator

$$X = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \phi_1(x,t,u,v)\frac{\partial}{\partial u} + \phi_2(x,t,u,v)\frac{\partial}{\partial v} \tag{31}$$

leaves invariant the set of solutions of (30). This yields to an overdetermined, nonlinear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\phi_1(x,t,u,v)$ and $\phi_2(x,t,u,v)$. We obtain classical potential symmetries if

$$(\xi_v)^2 + (\tau_v)^2 + (\phi_{1,v})^2 \neq 0. \tag{32}$$

The classical method applied to (30) leads to the classical symmetries.

5. Nonclassical potential symmetries

The basic idea for obtaining *nonclassical potential* symmetries is that the potential system (30) is augmented with the invariance surface conditions

$$\xi u_x + \tau u_t - \phi_1 = 0, \quad \xi v_x + \tau v_t - \phi_2 = 0, \tag{33}$$

which is associated with the vector field

$$X_1 = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \phi_1(x,t,u,v)\frac{\partial}{\partial u} + \phi_2(x,t,u,v)\frac{\partial}{\partial v}. \tag{34}$$

By requiring that both (30) and (33) are invariant under the transformations with infinitesimal generator (34) one obtains an overdetermined, nonlinear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\phi_1(x,t,u,v)$ and $\phi_2(x,t,u,v)$.

In the case $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,v) = 1$. The nonclassical method applied to (30) yields to the classical symmetries.

In the case $\tau = 0$, without loss of generality, we may set $\xi = 1$ and we obtain overdetermined nonlinear system of equations for the infinitesimals ϕ_1 and ϕ_2 which is solve by making ansatz on the form of $\phi_1(x,t,u,v)$ and $\phi_2(x,t,u,v)$. In this way we have found one solution.

For $a = 0$ and $m = -1$ we obtain the infinitesimal generators

$$\xi = 1, \quad \tau = 0, \quad \phi_1 = k\,u\,\psi(v), \quad \phi_2 = \omega(x,v),$$

where k is constant and ω and ψ satisfies $-k\psi\omega + \frac{\partial\omega}{\partial x} + \omega\frac{\partial\omega}{\partial v} = 0$.

In the case that $\omega = \omega(v)$ the infinitesimal generators are

$$\xi = 1, \quad \tau = 0, \quad \phi_1 = u\frac{d\omega}{dv}, \quad \phi_2 = \omega(v).$$

We obtain the nonclassical potential symmetry reduction

$$z = t, \quad u = \exp\left(kx\frac{d\omega}{dv}\right)h_1(t)$$

and v is given by $\int \frac{dv}{\omega(v)} = kx + h_2(t)$.

6. Hidden symmetries

We consider $m = 1$, then Eq. (2) is

$$u_{tt} - b\,u\,u_{xxxx} - 2\,b\,u_x u_{xxx} - b\,(u_{xx})^2 - 2\,u\,u_{xx} - a\,u_{xx} - 2\,(u_x)^2 = 0. \tag{35}$$

If we reduce Eq. (35) by using the generator $\lambda V_1 + V_2$ we get $u = h(z)$, $z = x - \lambda t$ and the reduced ODE is

$$b\,h\,h'''' + 2\,b\,h'h''' + b\,(h'')^2 + 2\,h\,h'' - \lambda^2 h'' + a\,h'' + 2\,(h')^2 = 0. \tag{36}$$

Applying the Lie classical method to Eq. (36) with $a = \lambda^2$ leads to a two-parameter Lie group.

$$X_1 = \frac{\partial}{\partial z}, \tag{37}$$

$$X_2 = h\frac{\partial}{\partial h}. \tag{38}$$

This symmetry is determined by a computer program such as SYM [8,9] or symgrp.max [7]. The inherited symmetry is $V_2 \to X_1$, which can be inferred by looking at the Lie algebra of the *Case 1*. The other symmetry is Type II hidden symmetries. The PDE from which the hidden symmetries are inherited is the original PDE in which we substitute the side condition from which the reduction has been derived

$$buu_{xxxx} + 2bu_x u_{xxx} + b(u_{xx})^2 + 2uu_{xx} + 2(u_x)^2 = 0. \tag{39}$$

We are going to derive some weak symmetries of the model equation (35), choosing as side condition the differential constraint

$$\lambda u_x + u_t = 0, \tag{40}$$

which is associated to the generator $V_1 + \lambda V_2$ that has been used to derive the reduction

$$h = u, \quad z = x - \lambda t.$$

Applying Lie classical method to the system (35) and (40) we get

$$\xi = F_1(t), \quad \tau = F_2(t), \quad \eta = F_3(t)u. \tag{41}$$

To apply the method in practice we use the MACSYMA package [7]. This yields the following generators:

$$U_1 = F_1(t)\frac{\partial}{\partial x}, \tag{42}$$

$$U_2 = F_2(t)\frac{\partial}{\partial t}, \tag{43}$$

$$U_3 = F_3(t)u\frac{\partial}{\partial u}, \tag{44}$$

where $F_i(t), i = 1, 2, 3$, are arbitrary functions. However, by appropriate choices of polynomials in t for $F_i(t)$ (and also taking combinations) the group generators reduce to the two generators (37,38). Consequently, we prove that X_2 is inherited as a weak symmetry of Eq. (36) with the side condition (40).

7. Concluding remarks

In this paper, the complete Lie group classification for a family Boussinesq equation (2) has been obtained. The corresponding reduced equations have been derived from the optimal system of subalgebras. We determine some nonclassical symmetries and some nonclassical potential symmetries for Eq. (2). We also have derived new travelling wave solutions. Among them we found solitons and compactons.

We also have obtained some Type-II hidden symmetries of a Boussinesq equation.

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