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Elimination orderings and localization in PBW algebras

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We characterize the existence of elimination orderings for a given PBW algebra. Elimination orderings on $\mathbb{N}^p$ are analyzed. A subclass of elimination orderings is considered to handle some Ore subsets and classical localizations.

1. Introduction and preliminaries

In this paper k is a (commutative) field of any characteristic. All rings are k-algebras. PBW algebras are defined in [1]. They are given by generators and quantum relations (definitions later), and they are also called polynomials of solvable type in [2] and G-algebras in [3–6]. In [7], there is an algorithm to check if a given algebra is a PBW algebra. The algorithm has two steps: first it computes an admissible ordering which "bounds" the quantum relations, and second Bergman's Diamond Lemma is used to check linear independence. However, the orderings computed in the first step are not suitable to handle subalgebras and eliminate variables. Let us define some concepts to clarify the previous ideas.

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Let R be a k-algebra and let $X = \{x_1, \ldots, x_p\}$ be a set of elements in R. Let $\mathbb{N}^p$ be the free abelian semigroup. For all $1 \leqslant i \leqslant p$ let ϵ_i be the canonical generators of $\mathbb{N}^p$, i.e. ϵ_i is the element of $\mathbb{N}^p$ such that all of its coordinates are equal to zero except the ith which is equal to 1.

(i) An admissible ordering $\preceq$ on $\mathbb{N}^p$ is a total order such that for all $\alpha, \beta, \gamma \in \mathbb{N}^p$, $\alpha \preceq \beta$ implies $\alpha + \gamma \preceq \beta + \gamma$. In commutative algebra they are also called monomial orderings since there is a closer connection with monomials in a commutative polynomial algebra.

(ii) Let $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p$. An element $X^\alpha = x_1^{\alpha_1} \cdots x_p^{\alpha_p}$ is called a *(standard) monomial in X*. A *(standard) polynomial in X* is a k–linear combination of standard monomials.

(iii) Let $f = \sum_\alpha c_\alpha X^\alpha$ be a polynomial in X. The *Newton diagram* of f is defined as $\mathcal{N}(f) = \{\alpha \in \mathbb{N}^p | c_\alpha \neq 0\}$. Let $\preceq$ be an admissible order in $\mathbb{N}^p$. The *exponent* of f (with respect to $\preceq$) is $\exp(f) = \max_\preceq \mathcal{N}(f)$.

(iv) A *quantum relation* is

$$x_j x_i = q_{ij} x_i x_j + p_{ij} \text{ where } q_{ij} \in k^* \text{ and } p_{ij} \text{ is a polynomial.} \quad (1)$$

This quantum relation is $\preceq$-bounded with respect to an admissible ordering $\preceq$ on $\mathbb{N}^p$ if and only if $\exp(p_{ij}) \prec \epsilon_i + \epsilon_j$. A full set of quantum relations is a set of quantum relations for each $1 \leqslant i < j \leqslant p$.

(v) R is said to be a PBW-algebra with respect to an admissible ordering $\preceq$ if

(PBW1) the set of standard monomials $\{X^\alpha | \alpha \in \mathbb{N}^p\}$ is a k-basis for R,
(PBW2) R satisfies a full set of $\preceq$-bounded quantum relations.

If R is a PBW-algebra then for all $\alpha, \beta \in \mathbb{N}^p$

$$X^\alpha X^\beta = q_{\alpha,\beta} X^{\alpha+\beta} + p_{\alpha,\beta} \quad \text{where } q_{\alpha,\beta} \in k^* \text{ and } \exp(p_{\alpha,\beta}) \prec \alpha + \beta \quad (2)$$

or equivalently for all $f, g \in R$

$$\exp(fg) = \exp(f) + \exp(g). \quad (3)$$

This can be seen in [1, Propositions 1.3 and 1.7], although [7] contains a more general approach. Let us see some examples.

Example 1. The quantum space $\mathcal{O}_q(k^p)$. It is generated by $X = \{x_1, \ldots, x_p\}$ and it satisfies the relations

$$x_j x_i = q_{ij} x_i x_j \quad \text{for all } 1 \leqslant i < j \leqslant p, \text{ where } q_{ij} \in k^*.$$

The commutative polynomial ring is a particular case of this quantum space when $q_{ij} = 1$ for all $i < j$.

Example 2. The $n \times n$ quantized uniparametric matrix algebra $\mathcal{O}_q(M_n(k))$ is generated by x_{ij}, $1 \leqslant i,j \leqslant n$ with relations

$$x_{ij} x_{kl} = \begin{cases} q x_{kl} x_{ij} & (k < i, j = l), \\ q x_{kl} x_{ij} & (k = i, j < l), \\ x_{kl} x_{ij} & (k < i, j > l), \\ x_{kl} x_{ij} + (q + q^{-1}) x_{kj} x_{il} & (k < i, l < j). \end{cases}$$

Example 3. Weyl algebras and enveloping algebras of finite-dimensional Lie algebras are also examples where $q_{ij} = 1$ for all $i < j$.

Example 4. Let $U = U_q(C)$ be the quantum enveloping algebra in the sense of [8,9] associated to a Cartan matrix C. This is an algebra over $\mathbb{C}(q)$, where q is an indeterminate. Following [10], U can be presented as a quotient of a PBW algebra. Details when $C = A_2$ are given in Appendix.

Let R be a k-algebra generated by $X = \{x_1, \ldots, x_p\}$ and satisfying a quantum relation (1) for each pair $1 \leqslant i < j \leqslant p$. In [7], an algorithm to check whenever R is a PBW-algebra with respect to X is provided. This algorithm is organized in two steps,

STEP 1 an admissible ordering $\preceq$ such that $\exp(p_{ij}) \prec \epsilon_i + \epsilon_j$ for all $i < j$ is computed if it exists [11],
STEP 2 Bergman's Diamond Lemma [12] is applied to see the linear independence of standard monomials.

A different but closer approach to the second task is the Non Degeneracy Condition studied in [5].

The computed ordering in the first step is a weighted one; a (weights) vector $\omega \in (\mathbb{N}^+)^p$ is computed such that for all $1 \leqslant i < j \leqslant p$ and all $\alpha \in \mathcal{N}(p_{ij})$, $\langle \alpha, \omega \rangle < \omega_i + \omega_j$; hence the ordering $\preceq_{lex_\omega}$ defined by

$$\alpha \preceq_{lex_\omega} \beta \iff \begin{cases} \langle \alpha, \omega \rangle < \langle \beta, \omega \rangle & \text{or} \\ \langle \alpha, \omega \rangle = \langle \beta, \omega \rangle \text{ and } \alpha \preceq_{lex} \beta \end{cases} \tag{4}$$

satisfies the requirements. Note that the lexicographical ordering $\preceq_{lex}$ can be replaced by any other admissible ordering.

These weighted orderings have been used to compute Gelfand–Kirillov dimension of finitely generated R-modules. They are useful for this task because all coordinates of ω are strictly positive. However they do not work properly to handle subalgebras. The elimination orderings are the right choice for this.

Let R be a k-algebra generated by $X = \{x_1, \ldots, x_p\}$ and satisfying a full set of quantum relations like (1). Let $\Sigma = \{x_{i_1}, \ldots, x_{i_r}\}$ be a subset of X and let $Y = X \setminus \Sigma$ be the complementary subset. We will identify Σ (resp. $X \setminus \Sigma$) with the subset $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, p\}$ (resp. $\{1, \ldots, p\} \setminus \{i_1, \ldots, i_r\}$). Let R_Σ be the k-subvector space of R generated by $\{\Sigma^\alpha = x_{i_1}^{\alpha_1} \cdots x_{i_r}^{\alpha_r} | \alpha \in \mathbb{N}^r\}$.

Let us define the canonical maps, injection

$$i_\Sigma : \mathbb{N}^r \longrightarrow \mathbb{N}^p; \quad \alpha \longmapsto i_\Sigma(\alpha) = \exp(\Sigma^\alpha) = \alpha_1 \epsilon_{i_1} + \cdots + \alpha_r \epsilon_{i_r}$$

and projection

$$\pi_\Sigma : \mathbb{N}^p \longrightarrow \mathbb{N}^r; \qquad \alpha \longmapsto \pi_\Sigma(\alpha) = (\alpha_{i_1}, \ldots, \alpha_{i_r}) = \alpha_{i_1} \epsilon_1 + \cdots + \alpha_{i_r} \epsilon_r.$$

Let $\mathbb{N}^r_\Sigma$ be the image of $\mathbb{N}^r$ in $\mathbb{N}^p$ via i_Σ. These definitions can be extended to Y if needed.

Definition 1 [6, Definition 5]. Let R be a PBW-algebra with respect to $X = \{x_1, \ldots, x_p\}$ and an admissible ordering $\preceq$. Let $\Sigma = \{x_{i_1}, \ldots, x_{i_r}\}$ be a subset of X. The ordering $\preceq$ is called an elimination ordering for $Y = X \setminus \Sigma$ if for any $f \in R$, $\exp(f) \in \mathbb{N}^r_\Sigma$ implies $f \in R_\Sigma$.

The next proposition is a direct consequence of the definition.

Proposition 2. *$\preceq$ is an elimination ordering for $X \setminus \Sigma$ if and only if for all $\alpha, \beta \in \mathbb{N}^p$, $\beta \in \mathbb{N}^r_\Sigma$ and $\alpha \preceq \beta$ imply $\alpha \in \mathbb{N}^r_\Sigma$.*

Proposition 3. *If $\preceq$ is an elimination ordering for $X \setminus \Sigma$ then*

(1) *for all $\{i, j\} \subseteq \{i_1, \ldots, i_r\}$, with $i < j$, $p_{ij} \in R_\Sigma$,*
(2) *R_Σ is a subalgebra of R.*

Proof. Since $p_{ij} = x_j x_i - q_{ij} x_i x_j$ for each pair $i < j$, the first assertion follows from the second one. So let $\alpha, \beta \in \mathbb{N}^r$. By (2) $\exp(\Sigma^\alpha \Sigma^\beta) = i_\Sigma(\alpha) + i_\Sigma(\beta) \in \mathbb{N}^r_\Sigma$, hence $\Sigma^\alpha \Sigma^\beta \in R_\Sigma$ as desired. □

Example 5. An Ore extension $A[x; \Sigma, \delta]$ of a ring A, where Σ is an automorphism of A and δ a Σ-derivation, is given by the rule $xa = \Sigma(a)x + \delta(a)$. Weyl algebras, quantum matrices and quantum spaces are instances of an iterated Ore extension $R = k[x_1][x_2; \Sigma_2, \delta_2] \cdots [x_p; \Sigma_p, \delta_p]$, where $\Sigma_j(x_i) = q_{ij} x_i$ for all $i < j$. The lexicographical ordering with $\epsilon_1 < \cdots < \epsilon_p$ is an admissible ordering for R. This ordering in an elimination ordering for all $\Sigma = \{x_1, \ldots, x_r\}$ and $Y = X \setminus \Sigma = \{x_{r+1}, \ldots, x_p\}$. See [2].

The existence of elimination orderings for a given PBW algebra is characterized in this paper. We compute a weights vector ω such that $\omega_i = 0$ for all index i such that $x_i \in \Sigma$ if an elimination ordering exists for $X \setminus \Sigma$. In order to prove this result, we have analyzed the elimination orderings on $\mathbb{N}^p$. Finally, a subclass of elimination orderings is considered to handle some Ore subsets and classical localizations.

2. Existence of elimination orderings

Let $p = e + r$ and let $\Sigma = \{e+1, \ldots, e+r\} \subseteq \{1, \ldots, p\}$. We say that an elimination ordering for $X \setminus \Sigma$ is an (e,r)-elimination ordering. We also denote $\mathrm{i}_\Sigma = i_r$ and $\pi_\Sigma = \pi_r$. Sometimes we will identify $\alpha^r = i_r\pi_r(\alpha)$ and $\alpha^e = \alpha - \alpha^r$. It follows from Proposition 2 that $\preceq$ is called an (e,r)-elimination ordering if and only if for all $\alpha \in \mathbb{N}^r$, and all $\beta \in \mathbb{N}^p$, $\beta \preceq i_r(\alpha)$ implies $\beta = \beta^r$. The next characterization allows the extension of the concept of elimination ordering.

Proposition 4. *$\preceq$ is an (e,r)-elimination ordering if and only if for all $1 \leqslant i \leqslant e$ and all $\alpha \in \mathbb{N}^r$, it follows $i_r(\alpha) \prec \epsilon_i$.*

Proof. Let $1 \leqslant i \leqslant e$ and $\alpha \in \mathbb{N}^r$. If $\epsilon_i \preceq i_r(\alpha)$ then $\epsilon_i = \epsilon_i^r$. But $\epsilon_i^r = 0$, hence $i_r(\alpha) \prec \epsilon_i$.

Conversely, let $\alpha \in \mathbb{N}^r$, $\beta \in \mathbb{N}^p$ such that $\beta \preceq i_r(\alpha)$. If $\beta \neq \beta^r$ then there exists $1 \leqslant i \leqslant e$ and $\beta' \in \mathbb{N}^p$ such that $\beta = \epsilon_i + \beta'$. So $\beta \preceq i_r(\alpha) \prec \epsilon_i \preceq \epsilon_i + \beta' = \beta$, a contradiction. □

Remark 5. Proposition 4 allows the extension of the definition of elimination orderings to $\mathbb{Z}^p$, $\mathbb{Q}^p$ and $\mathbb{R}^p$.

Definition 6. An admissible ordering $\preceq$ on $\mathbb{Z}^p$ is called (e,r)-elimination ordering if for all $1 \leqslant i \leqslant e$ and all $\alpha \in \mathbb{Z}^p$ (resp. $\alpha \in \mathbb{Z}^r$), it follows $\alpha^r \prec \epsilon_i$ (resp. $i_r(\alpha) \prec \epsilon_i$).

Proposition 7. *$\preceq$ is an (e,r)-elimination ordering on $\mathbb{N}^p$ if and only if its extension to $\mathbb{Z}^p$ is an (e,r)-elimination ordering and $\mathbb{N}^p$ is in the positive cone.*

Lemma 8. *Let $\preceq$ be an elimination ordering and let $0 \neq \omega \in (\mathbb{R}_0^+)^p$ such that $\beta \preceq \alpha$ implies $\langle \omega, \beta \rangle \leqslant \langle \omega, \alpha \rangle$. Then $\omega = (\omega_e, 0)$.*

Proof. Since $\preceq$ is an elimination ordering, for all $1 \leqslant i \leqslant e$ and all $\alpha \in \mathbb{N}^r$, $(0,\alpha) \preceq (\epsilon_i, 0)$. Hence, $\omega_i = \langle \epsilon_i, \omega \rangle \geqslant \langle (0,\alpha), \omega \rangle = \langle \alpha, \omega^r \rangle$. Let $s \in \mathbb{N}$ such that $s > \omega_i$ for all $1 \leqslant i \leqslant e$. If $\omega^r \neq 0$, then $\langle \omega^r, (s, \overset{r}{\ldots}, s) \rangle \geqslant s > \omega_i$, a contradiction. □

This lemma gives us how elimination orderings should look like. In [13], admissible orderings on $\mathbb{N}^p$ are parameterized by equivalence classes of matrices. For each ordering $\preceq$ there exists $A \in \mathscr{M}_{p\times m}(\mathbb{R})$ such that $\alpha \preceq \beta$ if and only if $\alpha A \preceq_{\mathrm{lex}} \beta A$; m depends on the dimension of entries of the matrix A as $\mathbb{Q}$-vector space; see [13] for details. In most examples $m = p$ and A is regular. The previous lemma says that the first column ω of A should satisfy $\omega = (\omega_e, 0)$. We can look for examples with this property.

Definition 9. Let $\preceq$ be an admissible ordering on $\mathbb{Z}^p$. A subset $\mathscr{B}$ of $\mathbb{Z}^p$ is called positive with respect to $\preceq$ if $\beta \succ 0$ for all $\beta \in \mathscr{B}$. This is denoted by $\mathscr{B} \succ 0$. $\mathscr{B}$ is called positive if there exists an admissible ordering such that $\mathscr{B}$ is positive with respect to it.

We want to characterize those finite subsets of $\mathbb{Z}^p$ such that they are positive with respect to an elimination ordering. The existence of elimination orderings is going to be reduced to the existence of $(e,1)$-elimination orderings.

Proposition 10. *Let $\mathscr{B}$ be a finite subset of $\mathbb{Z}^{e+1}$. If $\mathscr{B}$ is positive with respect to an $(e,1)$-elimination ordering $\preceq$ on $\mathbb{Z}^{e+1}$ then there exists $\omega \in \mathbb{R}^{e+1}$ such that*

(a) $\omega_1, \ldots, \omega_e > 0$,
(b) $\omega_{e+1} = 0$,
(c) $\langle \beta, \omega \rangle \geqslant 0$ *for all* $\beta \in \mathscr{B}$.

We need some previous lemmas to prove Proposition 10. So let $\mathscr{B}$ be a positive finite subset of $\mathbb{Z}^{e+1}$ and let $\preceq$ be an $(e, 1)$-elimination ordering. We assume $\epsilon_1 \succ \cdots \succ \epsilon_e \succ \epsilon_{e+1}$ without loss of generality. Let

$$\mathscr{B}' = \mathscr{B} \cup \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{e-1} - \epsilon_e, \epsilon_{e+1}\}$$

and for all $n \in \mathbb{N}$

$$\mathscr{B}_n = \mathscr{B}' \cup \{\epsilon_e - n\epsilon_{e+1}\}.$$

Since $\preceq$ is $(e, 1)$-elimination we have that $\mathscr{B}_n \succ 0$ for all $n \in \mathbb{N}$. Let $\mathscr{B}' = \{\beta^1, \ldots, \beta^s\}$. We can assume $\beta^j = \epsilon_j - \epsilon_{j+1}$ for all $1 \leqslant j \leqslant e-1$, and $\beta^e = \epsilon_{e+1}$. Let $r_1, \ldots, r_s \in \mathbb{R}^+ \setminus \mathbb{Q}$ such that $\{r_1, \ldots, r_s\}$ are $\mathbb{Q}$-linearly independent. For a fixed $n \in \mathbb{N}$ let us write $\beta^0 = \epsilon_e - n\epsilon_{e+1}$ and $r_0 = 1$. Let us call

$$C_n = \{\omega \in \mathbb{R}^{e+1} | \langle \beta^i, \omega \rangle \geqslant r_i \text{ for all } 0 \leqslant i \leqslant s\}.$$

Lemma 11. *C_n is a nonempty polytope.*

Proof. It is a polytope because it is the intersection of a finite number of hyperplanes. Let

$$O_n = \{\omega \in \mathbb{R}^{e+1} | \langle \beta, \omega \rangle > 0 \, \forall \beta \in \mathscr{B}_n\}.$$

It is clear that $C_n \subseteq O_n$. By [11, Proposition 2.1] O_n is nonempty, so let $\omega_0 \in O_n$ and let

$$\lambda_0 = \max_{0 \leqslant i \leqslant s} \left\{ \frac{r_i}{\langle \beta^i, \omega_0 \rangle} \right\}.$$

Then for all $0 \leqslant i \leqslant s$ we have $\langle \beta^i, \lambda_0 \omega_0 \rangle \geqslant r_i$, hence $\lambda_0 \omega_0 \in C_n$ and so C_n is nonempty. □

Lemma 12. *There exists $K \in \mathbb{R}^+$ such that for each vertex ω in one of the polytopes C_n and for all $1 \leqslant i \leqslant e-1$, we have $\omega_e \leqslant \omega_i \leqslant K\omega_e$.*

Proof. Since $\{\beta^0 = \epsilon_e - n\epsilon_{e+1}, \beta^1 = \epsilon_1 - \epsilon_2, \ldots, \beta^{e-1} = \epsilon_{e-1} - \epsilon_e, \beta^e = \epsilon_{e+1}\} \subseteq \mathscr{B}_n$, we have

$$\omega_1 \geqslant \cdots \geqslant \omega_e \geqslant \omega_{e+1} \geqslant r_e > 0. \tag{5}$$

Therefore, we only have to prove that $\omega_1 \leqslant K\omega_e$ for some $K \in \mathbb{R}^+$. Let ω be such a vertex. There exist $n \in \mathbb{N}$ and $e + 1$ elements $\{\beta^{i_1}, \ldots, \beta^{i_{e+1}}\} \subseteq \mathscr{B}_n$ such that ω is the unique solution of the linear system

$$\begin{aligned}
&\beta_1^{i_1} \omega_1 + \cdots + \beta_e^{i_1} \omega_e + \beta_{e+1}^{i_1} \omega_{e+1} = r_{i_1}, \\
&\quad\vdots \\
&\beta_1^{i_e} \omega_1 + \cdots + \beta_e^{i_e} \omega_e + \beta_{e+1}^{i_e} \omega_{e+1} = r_{i_e}, \\
&\beta_1^{i_{e+1}} \omega_1 + \cdots + \beta_e^{i_{e+1}} \omega_e + \beta_{e+1}^{i_{e+1}} \omega_{e+1} = r_{i_{e+1}}.
\end{aligned}$$

If $1 \leqslant i_1, \ldots, i_{e+1} \leqslant s$ then ω is independent of n and $\omega_i > 0$ for all i implies ω_1 is bounded by some positive multiple of ω_e. So let us assume without loss of generality that $i_{e+1} = 0$, i.e. the defining equations of ω are

$$\begin{aligned}
&\beta_1^{i_1} \omega_1 + \cdots + \beta_e^{i_1} \omega_e + \beta_{e+1}^{i_1} \omega_{e+1} = r_{i_1}, \\
&\quad\vdots \\
&\beta_1^{i_e} \omega_1 + \cdots + \beta_e^{i_e} \omega_e + \beta_{e+1}^{i_e} \omega_{e+1} = r_{i_e}, \\
&\omega_e - n\omega_{e+1} = 1.
\end{aligned}$$

Using Cramer's rule

$$\frac{\omega_1}{\omega_e} = \frac{K_1 + K_2 + K_3 n}{K_4 + K_5 n}, \text{ where } K_5 = \begin{vmatrix} \beta_1^{i_1} & \dots & \beta_{e-1}^{i_1} & r_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ \beta_1^{i_{e-1}} & \dots & \beta_{e-1}^{i_{e-1}} & r_{i_{e-1}} \\ \beta_1^{i_e} & \dots & \beta_{e-1}^{i_e} & r_{i_e} \end{vmatrix}.$$

Hence, if $K_5 \neq 0$ then $\frac{\omega_1}{\omega_e}$ is bounded as desired. Let

$$B = \begin{pmatrix} \beta_1^{i_1} & \dots & \beta_{e-1}^{i_1} & \beta_e^{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ \beta_1^{i_{e-1}} & \dots & \beta_{e-1}^{i_{e-1}} & \beta_e^{i_{e-1}} \\ \beta_1^{i_e} & \dots & \beta_{e-1}^{i_e} & \beta_e^{i_e} \end{pmatrix}$$

and let $B_{i,j}$ be the i,j-adjoint. Since $\beta^{i_1}, \dots, \beta^{i_e}$ are linearly independent it follows that the adjoints $B_{1,e}, \dots, B_{e-1,e}, B_{e,e}$ cannot be all of them equal to zero. But $r_{i_1} B_{1,e} + \dots + r_{i_{e-1}} B_{e-1,e} + r_{i_e} B_{e,e} = 0$ if and only if $B_{1,e} = \dots = B_{e-1,e} = B_{e,e} = 0$ since $\{r_{i_1}, \dots, r_{i_{e-1}}, r_{i_e}\}$ are $\mathbb{Q}$-linearly independent. Therefore, $K_5 \neq 0$ and the lemma is proved. $\square$

Proof [Proof of Proposition 10]. For all $n \in \mathbb{N}$ let ν^n be a vertex of the polytope (it exists by Lemma 11). Since $\nu_e^n \geqslant n\nu_{e+1}^n > 0$, we can define $\omega^n = \frac{1}{\nu_e^n}\nu^n$. By Lemma 12 and (5), and since $\nu_e^n \geqslant n\nu_{e+1}^n$, it follows:

$$1 \leqslant \omega_i^n = \frac{\nu_i^n}{\nu_e^n} \leqslant K \text{ for all } 1 \leqslant i \leqslant e \quad \text{and} \quad 0 < \omega_{e+1}^n = \frac{\nu_{e+1}^n}{\nu_e^n} \leqslant \frac{1}{n}. \tag{6}$$

(Note that $\omega_e^n = 1$ for all n). Let $\Sigma : \mathbb{N} \to \mathbb{N}$ be an increasing map such that all for all $1 \leqslant i \leqslant e+1$ the sequences $\{\omega_i^{\Sigma(n)}\}$ are convergent. Let

$$\nu_i = \lim_{n\to\infty} \omega_i^{\Sigma(n)} \text{ for all } 1 \leqslant i \leqslant e+1.$$

By (6) $\nu_1, \dots, \nu_e > 0$ and $\nu_{e+1} = 0$. For all $\beta \in \mathscr{B}$, since $\beta_1\nu_1^n + \dots + \beta^e\nu_e^n + \beta_{e+1}\nu_{e+1}^n > 0$, it follows:

$$\langle \beta, \nu \rangle = \beta_1\nu_1 + \dots + \beta_e\nu_e + \beta_{e+1}\nu_{e+1} = \lim_{n\to\infty} \beta_1\nu_1^n + \dots + \beta^e\nu_e^n + \beta_{e+1}\nu_{e+1}^n \geqslant 0.$$

So ν is the desired vector. $\square$

It remains to see the general situation. First we see the density results.

Lemma 13. *Let $A, B \subseteq \mathbb{Z}^p$ and let $X = \{x \in \mathbb{R}^p \setminus \{0\} | \forall\alpha \in A, \langle x, \alpha\rangle > 0 \text{ and } \forall\beta \in B, \langle x, \beta\rangle = 0\}$. If $X \neq \varnothing$ then $X \cap \mathbb{Q}^p \neq \varnothing$.*

Proof. Let $V = \{x \in \mathbb{R}^p \setminus \{0\} | \forall\beta \in B, \langle x, \beta\rangle = 0\}$ and $U = \{x \in \mathbb{R}^p \setminus \{0\} | \forall\alpha \in A, \langle x, \alpha\rangle > 0\}$. Then $X = U \cap V$ is a nonempty open set in V. Hence by density $X \cap \mathbb{Q}^p$ is also nonempty. $\square$

Proposition 14. *Let $\mathscr{B}$ be a finite subset of $\mathbb{Z}^p$. Let $H = \{x \in \mathbb{R}^p | x = x^e, \forall i \in \{1, \dots e\}, x_i > 0 \text{ and } \forall\beta \in \mathscr{B}, \langle x, \beta\rangle \geqslant 0\}$. If $H \neq \varnothing$ then $H \cap \mathbb{N}^p \neq \varnothing$.*

Proof. Fix $v \in H$. Let $A_1 = \{\beta \in \mathscr{B} | \langle v, \beta\rangle > 0\}$ and $B_1 = \{\beta \in \mathscr{B} | \langle v, \beta\rangle = 0\}$. Let $A = A_1 \cup \{\epsilon_i | 1 \leqslant i \leqslant e\}$ and $B = B_1 \cup \{\epsilon_i | e+1 \leqslant i \leqslant p\}$. Lemma 13 ensures $H \cap \mathbb{Q}^p \neq \varnothing$. Let $u \in H \cap \mathbb{Q}^p$, since $u_i \geqslant 0$ for all i, we can multiply by a common denominator to obtain $\omega \in H \cap \mathbb{N}^p$. $\square$

We can now prove the first theorem.

Theorem 15. *Let $\mathscr{B}$ be a positive finite subset of $\mathbb{Z}^p$ and let $\Sigma = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, p\}$. There exists an elimination ordering for $\{1, \ldots, p\} \setminus \Sigma$ on $\mathbb{Z}^p$ such that $\beta \succ 0$ for all $\beta \in \mathscr{B}$ if and only if there exists $\omega \in \mathbb{N}^p$ such that*

(a) $\omega_i \neq 0$ *for all* $i \notin \Sigma$,
(b) $\omega_j = 0$ *for all* $j \in \Sigma$,
(c) $\langle \beta, \omega \rangle \geqslant 0$ *for all* $\beta \in \mathscr{B}$.

Proof. Necessary condition is easy: Let $\preceq$ be the admissible ordering which allows $\mathscr{B}$ to be positive. If such vector ω exists then the ordering $\preceq_\omega$ defined by

$$\alpha \preceq_\omega \beta \iff \begin{cases} \langle \alpha, \omega \rangle < \langle \beta, \omega \rangle & \text{or} \\ \langle \alpha, \omega \rangle = \langle \beta, \omega \rangle & \text{and} \quad \alpha \preceq \beta \end{cases}$$

is an elimination ordering for $\{1, \ldots, p\} \setminus \Sigma$.

So let us prove sufficiency. Up to a reordering the positions we can assume $\Sigma = \{e+1, \ldots, e+r\}$, i.e. $\preceq$ is an (e, r)-elimination ordering. (This clarifies the notation). Let us define two maps.

$$\hat{}: \mathbb{Z}^{e+r} \longrightarrow \mathbb{Z}^{e+1},$$
$$\beta \longmapsto \hat{\beta} = (\beta_1, \ldots, \beta_e, \max\{\beta_{e+1}, \ldots, \beta_{e+r}\})$$

and

$$- : \mathbb{Z}^{e+1} \longrightarrow \mathbb{Z}^{e+r},$$
$$\beta \longmapsto \bar{\beta} = (\beta_1, \ldots, \beta_e, \beta_{e+1}, \overset{r}{\ldots}, \beta_{e+1}).$$

Let $\mathscr{B} = \{\beta^1, \ldots, \beta^s\} \subseteq \mathbb{Z}^{e+r}$ be positive and let $\preceq$ be an (e, r)-elimination ordering such that $\beta^i \succ 0$ for all $1 \leqslant i \leqslant s$. Let $\hat{\mathscr{B}} = \{\hat{\beta}^1, \ldots, \hat{\beta}^s\}$ and let $\preceq'$ be the ordering on $\mathbb{Z}^{e+1}$ defined by

$$\alpha \preceq' \beta \iff \bar{\alpha} \preceq \bar{\beta}.$$

It is easy to check that $\preceq'$ is an admissible ordering. Moreover, if $1 \leqslant i \leqslant e$ then $(0, \ldots, 0, n, \overset{r}{\ldots}, n) \prec \epsilon_i$ and hence $n\epsilon_{e+1} \prec' \epsilon_i$. Therefore $\preceq'$ is an $(e, 1)$-elimination ordering.

Since for all $\alpha \in \mathbb{Z}^{e+r}$ there exists $\gamma \in \mathbb{N}^{e+r}$ such that $\bar{\hat{\alpha}} = \alpha + \gamma$, it follows that $0 \prec \bar{\hat{\beta}}^i$ for all $1 \leqslant i \leqslant s$. Hence, $0 \prec' \hat{\beta}^i$ for all $1 \leqslant i \leqslant s$. By Proposition 10 there exists $\nu \in \mathbb{R}^{e+1}$ such that

(a) $\nu_1, \ldots, \nu_e > 0$,
(b) $\nu_{e+1} = 0$,
(c) $\langle \hat{\beta}, \nu \rangle \geqslant 0$ for all $\hat{\beta} \in \hat{\mathscr{B}}$.

Since $\bar{\nu} = (\nu_1, \ldots, \nu_e, 0, \overset{r}{\ldots}, 0)$ satisfies for all $\beta \in \mathscr{B}$

$$\langle \beta, \bar{\nu} \rangle = \langle \hat{\beta}, \nu \rangle \geqslant 0,$$

we have proven that $H = \{x \in \mathbb{R}^p | x_1, \ldots, x_e > 0, x_{e+1} = \cdots = x_{e+r} = 0 \text{ and } \forall \beta \in \mathscr{B}, \langle x, \beta \rangle \geqslant 0\} \neq \varnothing$, so by Proposition 14 there exists $\omega \in H \cap \mathbb{N}^p$, i.e. $\omega_1, \ldots, \omega_e > 0$, $\omega_{e+1} = \cdots = \omega_{e+r} = 0$ and for all $\beta \in \mathscr{B}$ $\langle \beta, \omega \rangle \geqslant 0$ as desired. □

3. Elimination of variables

Let R be an algebra generated by the set $X = \{x_1, \ldots, x_p\}$ and satisfying the quantum relations (1) for all $1 \leqslant i < j \leqslant p$.

Theorem 16. *Assume that there exists an admissible ordering $\preceq$ on $\mathbb{N}^p$ such that the quantum relations (1) are $\preceq$-bounded. Let $\Sigma = \{x_{i_1}, \ldots, x_{i_r}\}$ be a subset of X. The quantum relations (1) for all $1 \leqslant i < j \leqslant p$ of R are bounded with respect to an elimination ordering for $X \setminus \Sigma$ if and only if there exists $\omega \in \mathbb{N}^p$ such that*

(i) *for all $i \in \Sigma$, $\omega_i = 0$,*
(ii) *for all $i \in X \setminus \Sigma$, $\omega_i \neq 0$,*
(iii) *for all $i < j$ and all $\alpha \in \mathcal{N}(p_{ij})$, $\langle \alpha, \omega \rangle \leqslant \omega_i + \omega_j$.*

Proof. The relations of R are $\preceq$-bounded if and only if for all $1 \leqslant i < j \leqslant p$ and all $\alpha \in \mathcal{N}(p_{ij})$ it follows $\alpha \prec \epsilon_i + \epsilon_j$. Let $B_{ij} = \{\epsilon_i + \epsilon_j - \alpha | \alpha \in \mathcal{N}(p_{ij})\}$ and $\mathscr{B} = \bigcup_{i<j} B_{ij}$. Let us use the same symbol $\preceq$ to denote the extension of this ordering to $\mathbb{Z}^p$. Then the quantum relations of R are $\preceq$-bounded if and only if $\mathscr{B} \succ 0$. Hence, the proof follows from Theorem 15. □

This theorem can be used to decide effectively if there exists an elimination ordering for some variables in a PBW-algebra. Let R be an algebra generated by $X = \{x_1, \ldots, x_p\}$ and satisfying $\preceq$-quantum relations (1) for all $i < j$. Let $\Sigma = \{x_{i_1}, \ldots, x_{i_r}\}$. As in the previous proof let $B_{ij} = \{\epsilon_i + \epsilon_j - \alpha | \alpha \in \mathcal{N}(p_{ij})\}$ and $\mathscr{B} = \bigcup_{i<j} B_{ij}$.

Consider the following linear programming problem

$$\begin{aligned} &\textbf{minimize} \quad f(x_1, \ldots x_p) = x_1 + \cdots + x_p \\ &\textbf{with the constraints} \\ &\Phi \equiv \begin{cases} x_i \geqslant 1 & (i \notin \{i_1, \ldots, i_r\}), \\ x_j = 0 & (j \in \{i_1, \ldots, i_r\}), \\ \langle \beta, x \rangle \geqslant 0 & (\beta \in \mathscr{B}). \end{cases} \end{aligned} \tag{7}$$

Proposition 17. *The set of $\preceq$-bounded quantum relations (1) for all $i < j$ is $\preceq'$-bounded, where $\preceq'$ is an elimination ordering for $X \setminus \Sigma$, if and only if the linear programming problem (7) has a solution. Moreover for each solution ω of (7), the ordering $\preceq_\omega$ is an elimination ordering for $X \setminus \Sigma$.*

Proof. The linear programming problem (7) has a solution if and only if the feasible region Φ is not empty (notice that the linear functional $f(x_1, \ldots, x_p)$ is bounded from below whenever the feasible region is not empty). Hence, the proposition follows from Theorem 16. □

Remark 18. Thanks to Proposition 17 we can implement an algorithm to decide if there exists an elimination ordering for a given set of variables. It is desirable to choose an environment where the simplex algorithm is implemented. Once the simplex algorithm has provided a solution with real coordinates, the ideas in Lemma 13 and Proposition 14 allows to find a solution with non-negative integers as coordinates. In the examples we have checked, the solutions of the simplex algorithm are usually integer solutions as desired. The computations made in the appendix have been done with `Mathematica`©.

However, we think that Plural [14,4] is a very good choice. A procedure similar to `Gweights` in the library `nctools.lib` [15] can be developed. We do not expect any serious difficulties.

The main application of elimination orderings is the following classic result:

Proposition 19 [6, Lemma 2]. *Let R be a PBW algebra with respect to an elimination ordering $\preceq$ for Σ. Let $I \subseteq R$ be a left or a right ideal. If G is a Gröbner basis for I then $G \cap R_\Sigma$ is a Gröbner basis for $I \cap R_\Sigma$.*

So we can compute a set of generators for $I \cap R_\Sigma$ once we know a set of generators for I. In the commutative case, i.e. $R = k[x_1, \ldots, x_p]$, the thesis of Proposition 19 characterizes elimination orderings:

Lemma 20. *Let $R = k[x_1, \ldots, x_p]$, let $\preceq$ be an admissible ordering and $\Sigma \subseteq X$. If for each ideal $I \leqslant R$ and each Gröbner basis G of I, $G \cap R_\Sigma$ is a Gröbner basis for $I \cap R_\Sigma$ then $\preceq$ is an elimination ordering for $X \setminus \Sigma$.*

Proof. Let $\alpha \preceq \beta$ with $\beta \in \mathbb{N}^r_\Sigma$. Assume $\alpha \notin \mathbb{N}^r_\Sigma$. Let $G = \{X^\alpha, X^\beta\}$ and $G' = \{X^\alpha, X^\beta + X^\alpha\}$. It is clear that G and G' generate the same ideal I, and it is easy to see that G and G' are both Gröbner bases for I. Hence, $G \cap R_\Sigma = \{X^\beta\}$ and $G' \cap R_\Sigma = \varnothing$ are both Gröbner bases for $I \cap R_\Sigma$, a contradiction. Hence, $\alpha \in \mathbb{N}^r_\Sigma$ as desired. □

Remark 21. In view of Lemma 20, elimination orderings are necessary in the commutative setting to perform elimination of variables. In the non commutative setting R_Σ is not necessarily a subalgebra since it is the k-subspace of R generated by $\{\Sigma^\alpha | \alpha \in \mathbb{N}^r\}$, and G and G' are not necessarily a Gröbner basis for the left (or right) ideal they generate. So the previous proof does not work in the non commutative setting. However, since non commutative PBW algebras are more restrictive with respect to the possible orderings, it is not reasonable to think that elimination of variables can be performed without elimination orderings. In our setting we say that a set of variables can/cannot be eliminated with the help of Gröbner bases if there exists/there does not exist an elimination ordering for this set. If R_Σ is not a subalgebra we could consider the subalgebra generated by Σ, but we do not know how to work with this more general setting.

We finish this section with a result on the structure of PBW algebras with respect to an elimination ordering. Let $X = \{x_1, \ldots, x_p\}$ be a set of elements in R, $\Sigma = \{x_{i_1}, \ldots, x_{i_r}\}$ a subset as usual and $Y = X \setminus \Sigma = \{x_{j_1}, \ldots, x_{j_e}\}$, the complementary set. Let's regard that for all $\alpha \in \mathbb{N}^r$ and all $\beta \in \mathbb{N}^e$

$$\Sigma^\alpha = X^{i_\Sigma(\alpha)} \quad \text{and} \quad Y^\beta = X^{i_Y(\beta)}. \tag{8}$$

Proposition 22. *Assume R is a PBW algebra with respect to X and an elimination ordering $\preceq$ for Y. Then R is a left (right) free R_Σ-module with basis $\{Y^\beta | \beta \in \mathbb{N}^e\}$.*

Proof. Let first see that all $f \in R$ belongs to $\sum_{\beta \in \mathbb{N}^e} R_\Sigma Y^\beta$. We are going to prove it by induction on $\exp(f)$. If $\exp(f) = 0$ (even if $\exp(f) \in \mathbb{N}^r_\Sigma$) then the result is clear. So let $f = cX^{\exp(f)} + f'$ with $\exp(f') \prec \exp(f)$. Let's call $\alpha = \pi_\Sigma(\exp(f))$ and $\beta = \pi_Y(\exp(f))$, then $\exp(f) = i_\Sigma(\alpha) + i_Y(\beta)$. By (2) there exists $q = q_{\alpha,\beta} \in k^*$, such that

$$X^{i_\Sigma(\alpha)+i_Y(\beta)} = qX^{i_\Sigma(\alpha)}X^{i_Y(\beta)} + p, \text{ where } \exp(p) \prec i_\Sigma(\alpha) + i_Y(\beta).$$

Hence by (8)

$$f = cq\Sigma^\alpha Y^\beta + cp + f' \text{ with } \exp(cp + f') \prec \exp(f).$$

Induction hypothesis ensures $cp + f' \in \sum_{\gamma \in \mathbb{N}^e} R_\Sigma Y^\gamma$, and therefore $f \in \sum_{\gamma \in \mathbb{N}^e} R_\Sigma Y^\gamma$ as desired.

It remains to prove the linear independence. Consider the expression

$$f_1 Y^{\beta^1} + \cdots + f_t Y^{\beta^t}, \text{ where } f_1, \ldots, f_t \in R_\Sigma \setminus \{0\}.$$

We can assume $\beta^1 \prec \cdots \prec \beta^t$. Hence, $\exp(f_i Y^{\beta^i}) \neq \exp(f_j Y^{\beta^j})$ for all $1 \leqslant i \neq j \leqslant t$. Then $\exp(f_1 Y^{\beta^1} + \cdots + f_t Y^{\beta^t}) = \max\{\exp(f_i Y^{\beta^i}) | 1 \leqslant i \leqslant t\}$ and $f_1 Y^{\beta^1} + \cdots + f_t Y^{\beta^t} \neq 0$. The linear independence is proven. □

4. Block orderings and localization

Let us analyze the localization. To localize in Ore sets we are going to focus on an interesting family of examples of elimination orderings, the block orderings.

Definition 23. Let $\preceq_r$ and $\preceq_e$ be admissible orderings on $\mathbb{N}^r$ and $\mathbb{N}^e$, respectively. Let $p = e + r$ as usual. The ordering $\preceq$ on $\mathbb{N}^p$ defined by

$$\alpha \preceq \beta \iff \begin{cases} \pi_Y(\alpha) \prec_e \pi_Y(\beta) & \text{or} \\ \pi_Y(\alpha) = \pi_Y(\beta) & \text{and} \quad \pi_\Sigma(\alpha) \preceq_r \pi_\Sigma(\beta) \end{cases}$$

is called (Y, Σ)-block ordering.

Proposition 24. *A block ordering is an elimination ordering for $Y = X \setminus \Sigma$.*

Proof. Let $\beta \in \mathbb{N}^r_\Sigma$, i.e. $\beta = i_\Sigma(\pi_\Sigma(\beta))$, and let $\alpha \prec \beta$. Then $\pi_Y(\alpha) \preceq_e \pi_Y(\beta)$ since $\preceq$ is a block ordering. But $\pi_Y(\beta) = 0$ because $\beta \in \mathbb{N}^r_\Sigma$, hence $\pi_Y(\alpha) = 0$ and then $\alpha \in \mathbb{N}^r_\Sigma$ as desired. □

Remark 25. Let $\preceq$ be a (Y, Σ)-block ordering, where $\preceq_e$ and $\preceq_r$ are the corresponding admissible orderings on $\mathbb{N}^e$ and $\mathbb{N}^r$. It is easy to see that for all $\alpha, \beta \in \mathbb{N}^e$, $\alpha \prec_e \beta$ if and only if $i_Y(\alpha) \prec i_Y(\beta)$. The same applies for each pair $\gamma, \delta \in \mathbb{N}^r$. Hence, the orderings on each part of $\mathbb{N}^p$ can be recovered from $\preceq$.

Remark 26. It is well known that tensor products of PBW algebras are PBW algebras. The orderings which provide PBW structures on tensor products are block orderings. See [7] for details.

Example 6. Let $\preceq$ be the ordering on $\mathbb{N}^4$ defined by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

i.e. $\alpha \prec \beta$ if and only if

$$(\alpha_1 + 2\alpha_2, \alpha_3 + \alpha_4, \alpha_1 + 3\alpha_2 + \alpha_4, \alpha_3) \prec_{\text{lex}} (\beta_1 + 2\beta_2, \beta_3 + \beta_4, \beta_1 + 3\beta_2 + \beta_4, \beta_3).$$

Since the first column is $(1, 2, 0, 0)$, this ordering is a $(\{1, 2\}, \{3, 4\})$-elimination ordering. However, we have

$$(1, 1, 1, 0) \prec (3, 0, 1, 1)$$

but

$$(1, 1, 0, 0) \succ (3, 0, 0, 0),$$

hence $\preceq$ is not a $(\{1, 2\}, \{3, 4\})$-block ordering.

The existence of block orderings can be characterized in a similar way to Theorem 15:

Theorem 27. *Let $\mathscr{B}$ be a positive finite subset of $\mathbb{Z}^p$ and let $\Sigma = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, p\}$. There exists a block ordering for $Y = \{1, \ldots, p\} \setminus \Sigma$ on $\mathbb{Z}^p$ such that $\beta \succ 0$ for all $\beta \in \mathscr{B}$ if and only if there exist $\omega^e \in (\mathbb{N}^+)^e$ and $\omega^r \in (\mathbb{N}^+)^r$ such that*

$$\forall \beta \in \mathscr{B}, \quad \begin{cases} \langle \pi_Y(\beta), \omega^e \rangle > 0 & \text{or} \\ \langle \pi_Y(\beta), \omega^e \rangle = 0 & \text{and} \quad \langle \pi_\Sigma(\beta), \omega^r \rangle > 0. \end{cases}$$

Proof. As before, if there exist $\omega^e \in (\mathbb{N}^+)^e$ and $\omega^r \in (\mathbb{N}^+)^r$ satisfying the desired properties then the ordering

$$\alpha \preceq \beta \iff \begin{cases} \pi_Y(\alpha) \prec_{lex_{\omega^e}} \pi_Y(\beta) & \text{or} \\ \pi_Y(\alpha) = \pi_Y(\beta) & \text{and} \quad \pi_\Sigma(\alpha) \preceq_{lex_{\omega^r}} \pi_\Sigma(\beta) \end{cases}$$

is a block ordering such that $\beta \succ 0$ for all $\beta \in \mathscr{B}$. Recall that $\preceq_{lex_{\omega^e}}$ and $\preceq_{lex_{\omega^r}}$ are defined in (4).

So assume $\mathscr{B}$ is positive with respect to a block ordering $\preceq$, and let $\preceq_e$, $\preceq_r$ be the corresponding orderings on $\mathbb{N}^e$ and $\mathbb{N}^r$. Let $\mathscr{B}^e = \{\pi_Y(\beta) | \beta \in \mathscr{B}\} \setminus \{0\} \subseteq \mathbb{Z}^e$ and $\mathscr{B}^r = \{\pi_\Sigma(\beta) | \beta \in \mathscr{B}, \pi_Y(\beta) = 0\} \subseteq \mathbb{Z}^r$.

Since $\preceq$ is a block ordering we have $\mathscr{B}^e$, $\mathscr{B}^r$ are positive with respect to $\preceq_e$ and $\preceq_r$ respectively. By [11, Proposition 2.1]1006.16023, there exist $\omega^e \in (\mathbb{N}^+)^e$ and $\omega^r \in (\mathbb{N}^+)^r$ such that for all $\beta' \in \mathscr{B}^e$ $\langle \beta', \omega^e \rangle > 0$ and for all $\beta'' \in \mathscr{B}^r$ $\langle \beta'', \omega^r \rangle > 0$. Hence for all $\beta \in \mathscr{B}$, $\pi_Y(\beta) \in \mathscr{B}^e$ and $\langle \pi_Y(\beta), \omega^e \rangle > 0$, or $\pi_\Sigma(\beta) \in \mathscr{B}^r$ and $\langle \pi_\Sigma(\beta), \omega^r \rangle > 0$ as desired. □

Let R be a domain. Recall that $S \subseteq R$ is a left Ore set if for all $r \in R$ and all $s \in S$ there exist $r' \in R$ and $s' \in S$ such that $s'r = r's$. Right Ore sets are defined analogously. Ore sets are those which allow the definition of rings of quotients: fractions are equivalence classes of pairs $(s, a) \in S \times R$ via the equivalence relation $(s, a) \sim (t, b)$ if and only if there exist $c, d \in R$ such that $cs = dt \in S$ and $ca = db$. The arithmetic is defined by

$$(s, a) + (t, b) = (u, ca + db), \text{ where } u = cs = dt \in S,$$
$$(s, a) \cdot (t, b) = (us, cb), \text{ where } ua = ct \quad \text{and} \quad u \in S.$$

It is denoted by $Q_S(R)$. We refer to [16, Section 2.1] or [17] for details concerning localization. In particular it is proved that if R is a Noetherian domain, the set of nonzero elements is an Ore set. We use widely Proposition 22.

Proposition 28. *Let R be a PBW algebra with respect to X and a (Y, Σ)-block ordering $\preceq$. Let $S = R_\Sigma \setminus \{0\}$. Then S is a left and right Ore set in R.*

Proof. Let $s \in S$ and $\sum_{\delta \preceq \lambda} a_\delta Y^\delta \in R$. In order to prove that S is left Ore we are going to find $s' \in S$ and $\sum_{\delta \preceq \lambda'} a'_\delta Y^\delta \in R$ such that

$$\sum_{\delta \preceq \lambda'} a'_\delta Y^\delta s = s' \sum_{\delta \preceq \lambda} a_\delta Y^\delta. \tag{9}$$

Since $\preceq$ is a block ordering it follows that

$$\exp\left(s' \sum_{\delta \preceq \lambda} a_\delta Y^\delta \right) = \exp(s') + \exp(a_\lambda) + \mathrm{i}_Y(\lambda), \tag{10}$$

$$Y^\gamma s = \sum_{\delta \preceq \gamma} s_{\gamma\delta} Y^\delta, \text{ where } s_{\gamma\gamma} \in S. \tag{11}$$

It also follows from (10) that

$$\exp\left(\sum_{\delta \preceq \lambda'} a'_\delta Y^\delta s \right) = \exp(a_{\lambda'}) + \exp(s) + \mathrm{i}_Y(\lambda'). \tag{12}$$

So $\lambda' = \lambda$ if both s' and $\sum_{\delta \preceq \lambda'} a'_\delta Y^\delta$ exist. Since

$$s' \sum_{\delta \preceq \lambda} a_\delta Y^\delta = \sum_{\delta \preceq \lambda} (s' a_\delta) Y^\delta,$$

by (10)

$$\begin{aligned} \left(\sum_{\gamma \preceq \lambda} a'_\gamma Y^\gamma \right) s &= \sum_{\gamma \preceq \lambda} a'_\gamma (Y^\gamma s) \\ &= \sum_{\gamma \preceq \lambda} a'_\gamma \left(\sum_{\delta \preceq \gamma} s_{\gamma\delta} Y^\delta \right) \\ &= \sum_{\delta \preceq \gamma \preceq \lambda} a'_\gamma s_{\gamma\delta} Y^\delta. \end{aligned}$$

So for all $\delta \preceq \lambda$ we have to find s' and a'_γ, for all $\gamma \preceq \lambda$ such that

$$s'a_\delta = \sum_{\delta \preceq \gamma \preceq \lambda} a'_\gamma s_{\gamma\delta}. \tag{13}$$

Let us proceed by induction on λ. If $\lambda = 0$, then $\sum_{\delta \preceq \lambda} a_\delta Y^\delta = a_0 \in R_\Sigma$, and since $S \subseteq R_\Sigma$ is left Ore (see e.g. [16, 2.1.15]) then there exist $a' \in R_\Sigma$ and $s \in S$ such that $s'a = a's$. So assume $\lambda \succ 0$.

For all $\delta \preceq \lambda$ let $b_\delta \in S = R_\Sigma \setminus \{0\}$ such that

$$\begin{cases} s_{\lambda\delta} b_\delta = s_{\lambda\lambda} b_\lambda \in S & \text{if } s_{\lambda\delta} \neq 0, \\ b_\delta = 1 & \text{if } s_{\lambda\delta} = 0. \end{cases} \tag{14}$$

They exist because S is right Ore in the domain R_Σ, see [16, 2.1.8]. For all $\delta \prec \lambda$ let

$$\overline{a}_\delta = \begin{cases} a_\delta b_\delta - a_\lambda b_\lambda & \text{if } s_{\lambda\delta} \neq 0, \\ a_\delta b_\delta = a_\delta & \text{if } s_{\lambda\delta} = 0. \end{cases}$$

By induction there exist $\sum_{\gamma \prec \lambda} a''_\gamma Y^\gamma$ and s'' such that for all $\delta \prec \lambda$

$$s''\overline{a}_\delta = \sum_{\delta \preceq \gamma \prec \lambda} a''_\gamma s_{\gamma\delta} b_\delta. \tag{15}$$

Moreover there exist $a'''_\lambda \in R_\Sigma$ and $s''' \in S$ such that

$$a'''_\lambda s_{\lambda\lambda} b_\lambda = s''' a_\lambda b_\lambda \tag{16}$$

because S is left Ore in R_Σ. Let s^{IV} and s^V such that

$$s^{IV} s'' = s^V s''' = s' \in S \tag{17}$$

and let

$$a'_\delta = \begin{cases} s^{IV} a''_\delta & \text{if } \delta \prec \lambda, \\ s^V a'''_\lambda & \text{if } \delta = \lambda. \end{cases}$$

First

$$\begin{aligned} s'a_\lambda b_\lambda &= s^V s''' a_\lambda b_\lambda \quad \text{by (17)} \\ &= s^V a'''_\lambda s_{\lambda\lambda} b_\lambda \quad \text{by (16)} \\ &= a'_\lambda s_{\lambda\lambda} b_\lambda \text{ by definition,} \end{aligned}$$

hence, $s'a_\lambda = a'_\lambda s_{\lambda\lambda} = \sum_{\lambda \preceq \gamma \preceq \lambda} a'_\gamma s_{\gamma\lambda}$ because R_Σ is a domain, and (13) has solution in this case. Now assume $\delta \prec \lambda$ and $s_{\lambda\delta} = 0$. Then

$$\begin{aligned} s'a_\delta b_\delta &= s^{IV} s'' \overline{a}_\delta \quad \text{by (17) and definition} \\ &= \sum_{\delta \preceq \gamma \prec \lambda} s^{IV} a''_\gamma s_{\gamma\delta} b_\delta \text{ by (15)} \\ &= \sum_{\delta \preceq \gamma \prec \lambda} a'_\gamma s_{\gamma\delta} b_\delta \text{ by definition} \\ &= \sum_{\delta \preceq \gamma \preceq \lambda} a'_\gamma s_{\gamma\delta} b_\delta \text{ because } s_{\lambda\delta} = 0. \end{aligned}$$

We can also cancel b_δ out and we have (13) in this second case. Finally, let $\delta \prec \lambda$ and $s_{\lambda\delta} \neq 0$. We have

$$\begin{aligned} s'a_\delta b_\delta &= s'(\overline{a}_\delta + a_\lambda b_\lambda) \text{ by definition} \\ &= s^{IV} s'' \overline{a}_\delta + s^V s''' a_\lambda b_\lambda \text{ by (17)} \\ &= \sum_{\delta \preceq \gamma \prec \lambda} s^{IV} a''_\gamma s_{\gamma\delta} b_\delta + s^V s''' a_\lambda b_\lambda \text{ by (15)} \end{aligned}$$

$$= \sum_{\delta \preceq \gamma \prec \lambda} s^{IV} a''_{\gamma} s_{\gamma\delta} b_{\delta} + s^{V} a'''_{\lambda} s_{\lambda\lambda} b_{\lambda} \text{ by (16)}$$
$$= \sum_{\delta \preceq \gamma \prec \lambda} s^{IV} a''_{\gamma} s_{\gamma\delta} b_{\delta} + s^{V} a'''_{\lambda} s_{\lambda\delta} b_{\delta} \text{ by (14)}$$
$$= \sum_{\delta \preceq \gamma \preceq \lambda} a'_{\gamma} s_{\gamma\delta} b_{\delta} \text{ by definition.}$$

Once again we can cancel b_δ out and we have (13) in the lase case. Therefore we have covered all possible cases and the proof is finished. □

What can we say about effective computations? Using syzygy modules, if A is a PBW algebra and $S = A \setminus \{0\}$ then the localization $Q_S(A)$ is a computable division ring. See [18,7] for details (see [19] for a previous non commutative reference in enveloping algebras of Lie algebras). In our setting, we have to take more care. We recall from [20] that a ring extension $\Bbbk \subseteq B$ of a division ring $\Bbbk$ is called a PBW ring if there exists $X = \{x_1, \ldots, x_p\} \subseteq B$ and an admissible ordering $\preceq$ on $\mathbb{N}^p$ such that

(1) B is a free left $\Bbbk$-module with the standard monomials in X as a basis.
(2) for every $1 \leqslant i \leqslant p$ and every $a \in \Bbbk \setminus \{0\}$ there exist $q_{ia} \in \Bbbk \setminus \{0\}$ and a standard polynomial p_{ia} such that

$$x_i a = q_{ia} x_i + p_{ia} \quad \text{and} \quad \exp(p_{ia}) \prec \epsilon_i,$$

(3) for each $1 \leqslant i < j \leqslant p$ there exist $q_{ij} \in \Bbbk \setminus \{0\}$ and a standard polynomial p_{ij} such that

$$x_j x_i = q_{ij} x_i x_j + p_{ij} \quad \text{and} \quad \exp(p_{ij}) \prec \epsilon_i + \epsilon_j.$$

i.e. it satisfies a full set of quantum relations for a non commutative base ring. The arithmetic and algorithms to compute in a PBW ring can be seen in [20,7]. So effective computations are possible in PBW rings.

Theorem 29. *Let R be a PBW algebra with respect to a (Y, Σ)-block ordering $\preceq$ and let $S = R_\Sigma \setminus \{0\}$. Then $Q_S(R)$ is isomorphic to a PBW ring over the division ring $Q_S(R_\Sigma)$.*

Proof. Let T be the free left $Q_S(R_\Sigma)$-module with basis $\{Y^\alpha | \alpha \in \mathbb{N}^e\}$. Let Φ be the map

$$\Phi : Q_S(R) \longrightarrow T,$$
$$\left(s, \sum_\alpha a_\alpha Y^\alpha\right) \longmapsto \sum_\alpha (s, a_\alpha) Y^\alpha.$$

Let us prove that Φ is well defined. Assume $(s, \sum_\alpha a_\alpha Y^\alpha) \sim \left(t, \sum_\beta b_\beta Y^\beta\right)$. Then there exist $c, d \in R$ such that $cs = dt \in S$ and $c \sum_\alpha a_\alpha Y^\alpha = d \sum_\beta b_\beta Y^\beta$. Since $cs = dt \in S$ and $\preceq$ is a block ordering, it follows that $c, d \in S$. Hence, $\sum_\alpha c a_\alpha Y^\alpha = c \sum_\alpha a_\alpha Y^\alpha = d \sum_\beta b_\beta Y^\beta = \sum_\beta d b_\beta Y^\beta$ and then $\sum_\alpha (s, a_\alpha) Y^\alpha = \sum_\beta (t, b_\beta) Y^\beta$. If $\sum_\alpha (s, a_\alpha) Y^\alpha = 0$, then $a_\alpha = 0$ for all $\alpha \in \mathbb{N}^e$, hence Φ is injective. Moreover, given $\sum_\alpha (s_\alpha, a_\alpha) Y^\alpha \in T$ there exist $b_\alpha \in R_\Sigma$ for all $\alpha, \beta \in \mathbb{N}^e$ such that $a_\alpha, a_\beta \neq 0$ satisfying $b_\alpha s_\alpha = b_\beta s_\beta = s \in S$, so $(s_\alpha, a_\alpha) \sim (s, b_\alpha a_\alpha)$ and $\sum_\alpha (s_\alpha, a_\alpha) Y^\alpha = \Phi\left(s, \sum_\alpha b_\alpha a_\alpha Y^\alpha\right)$. Hence, Φ is also surjective. Via this bijection T is an algebra.

Let $(s, a) \in Q_S(R_\Sigma)$ and $y_i \in \{y_1, \ldots, y_e\}$. By Proposition 22, using the fact that $\preceq$ is a block ordering, $a y_i = q_{i,a} y_i + p_{i,a}$ where $q_{i,a} \in R_\Sigma$ and $p_{i,a} = \sum_{\alpha \prec_e \epsilon_i} a_\alpha Y^\alpha$. So

$$y_i (s, a) = (s, q_{i,a}) y_i + \sum_{\alpha \prec_e \epsilon_i} (s, a_\alpha) Y^\alpha.$$

Analogously

$$y_j y_i = q_{ij} y_i y_j + \sum_{\alpha \prec_e \epsilon_i + \epsilon_j} a_\alpha Y^\alpha.$$

Hence, T is a PBW ring over $Q_S(R_\Sigma)$. $\square$

Therefore, effective computations are possible in $Q_S(R)$ when R is a PBW algebra with respect to a (Y, Σ)-block ordering and $S = R_\Sigma \setminus \{0\}$.

Appendix: $U_q(A_2)$

We finish with a non trivial example, the quantized enveloping algebra associated to a Cartan matrix of type A_2. Consider the PBW algebra V generated by $f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2, e_{12}, e_{13}, e_{23}$ and satisfying the following relations:

$$\begin{aligned}
&e_{13}e_{12} = q^{-2}e_{12}e_{13}, && f_{13}f_{12} = q^{-2}f_{12}f_{13},\\
&e_{23}e_{12} = q^{2}e_{12}e_{23} - qe_{13}, && f_{23}f_{12} = q^{2}f_{12}f_{23} - qf_{13},\\
&e_{23}e_{13} = q^{-2}e_{13}e_{23}, && f_{23}f_{13} = q^{-2}f_{13}f_{23},
\end{aligned}$$

$$\begin{aligned}
& & &e_{12}k_1 = q^{-2}k_1e_{12}, && k_1f_{12} = q^{-2}f_{12}k_1,\\
&e_{12}f_{12} = f_{12}e_{12} + \tfrac{k_1^2 - l_1^2}{q^2 - q^{-2}}, & &e_{12}k_2 = qk_2e_{12}, && k_2f_{12} = qf_{12}k_2,\\
&e_{12}f_{13} = f_{13}e_{12} + qf_{23}k_1^2, & &e_{13}k_1 = q^{-1}k_1e_{13}, && k_1f_{13} = q^{-1}f_{13}k_1,\\
&e_{12}f_{23} = f_{23}e_{12}, & &e_{13}k_2 = q^{-1}k_2e_{13}, && k_2f_{13} = q^{-1}f_{13}k_2,\\
&e_{13}f_{12} = f_{12}e_{13} - q^{-1}l_1^2e_{23}, & &e_{23}k_1 = qk_1e_{23}, && k_1f_{23} = qf_{23}k_1,\\
&e_{13}f_{13} = f_{13}e_{13} - \tfrac{k_1^2k_2^2 - l_1^2l_2^2}{q^2 - q^{-2}}, & &e_{23}k_2 = q^{-2}k_2e_{23}, && k_2f_{23} = q^{-2}f_{23}k_2,\\
& & &e_{12}l_1 = q^{2}l_1e_{12}, && l_1f_{12} = q^{2}f_{12}l_1,\\
&e_{13}f_{23} = f_{23}e_{13} + qk_2^2e_{12}, & &e_{12}l_2 = q^{-1}l_2e_{12}, && l_2f_{12} = q^{-1}f_{12}l_2,\\
&e_{23}f_{12} = f_{12}e_{23}, & &e_{13}l_1 = ql_1e_{13}, && l_1f_{13} = qf_{13}l_1,\\
&e_{23}f_{13} = f_{13}e_{23} - q^{-1}f_{12}l_2^2, & &e_{13}l_2 = ql_2e_{13}, && l_2f_{13} = qf_{13}l_2,\\
&e_{23}f_{23} = f_{23}e_{23} + \tfrac{k_2^2 - l_2^2}{q^2 - q^{-2}}, & &e_{23}l_1 = q^{-1}l_1e_{23}, && l_1f_{23} = q^{-1}f_{23}l_1,\\
& & &e_{23}l_2 = q^{2}l_2e_{23}, && l_2f_{23} = q^{2}f_{23}l_2,
\end{aligned}$$

$$\begin{aligned}
&l_1k_1 = k_1l_1, && l_2k_1 = k_1l_2, && k_2k_1 = k_1k_2,\\
&l_1k_2 = k_2l_1, && l_2k_2 = k_2l_2, && l_2l_1 = l_1l_2.
\end{aligned}$$

The elements $k_1l_1 - 1$ and $k_2l_2 - 1$ are in the center of V, so the left (or right) ideal I generated by them is twosided. The quantized enveloping algebra associated to A_2 is $U_q(A_2) = V/I$. This follows from [21, Section 3]. The constraints associated to the set $\mathscr{B}$ in (7) are (see also [11, Appendix])

$$\begin{aligned}
&\mathtt{f}_{12} - \mathtt{f}_{13} + \mathtt{f}_{23} \geqslant 0, && \mathtt{e}_{12} - \mathtt{e}_{13} + \mathtt{e}_{23} \geqslant 0,\\
&-2\mathtt{k}_1 - \mathtt{f}_{23} + \mathtt{e}_{12} + \mathtt{f}_{13} \geqslant 0, && -2\mathtt{k}_2 - \mathtt{e}_{12} + \mathtt{e}_{13} + \mathtt{f}_{23} \geqslant 0,\\
&-2\mathtt{l}_1 - \mathtt{e}_{23} + \mathtt{e}_{13} + \mathtt{f}_{12} \geqslant 0, && -\mathtt{f}_{12} - 2\mathtt{l}_2 + \mathtt{e}_{23} + \mathtt{f}_{13} \geqslant 0,\\
&-2\mathtt{k}_1 + \mathtt{e}_{12} + \mathtt{f}_{12} \geqslant 0, && -2\mathtt{l}_1 + \mathtt{e}_{12} + \mathtt{f}_{12} \geqslant 0,\\
&-2\mathtt{k}_1 - 2\mathtt{k}_2 + \mathtt{e}_{13} + \mathtt{f}_{13} \geqslant 0, && -2\mathtt{l}_1 - 2\mathtt{l}_2 + \mathtt{e}_{13} + \mathtt{f}_{13} \geqslant 0,\\
&-2\mathtt{k}_2 + \mathtt{e}_{23} + \mathtt{f}_{23} \geqslant 0, && -2\mathtt{l}_2 + \mathtt{e}_{23} + \mathtt{f}_{23} \geqslant 0.
\end{aligned}$$

The other constraints depend on the variables we want to eliminate. For instance:

(1) If we want to check that the variables f_{13}, f_{23} can be eliminated, then we have to add these constraints

$$\begin{aligned}
&\mathtt{f}_{12} = 0, \quad \mathtt{f}_{13} \geqslant 1, \quad \mathtt{f}_{23} \geqslant 1,\\
&\mathtt{k}_1 = 0, \quad \mathtt{k}_2 = 0, \quad \mathtt{l}_1 = 0, \quad \mathtt{l}_2 = 0,\\
&\mathtt{e}_{12} = 0, \quad \mathtt{e}_{13} = 0, \quad \mathtt{e}_{23} = 0.
\end{aligned}$$

The associated linear programming problem gives as solution $\mathtt{f}_{13} = 1$ and $\mathtt{f}_{23} = 1$, so these variables can be eliminated.

(2) If we consider the variables f_{12} and f_{23}, the constraints are

$$\mathtt{f}_{12} \geqslant 1, \quad \mathtt{f}_{13} = 0, \quad \mathtt{f}_{23} \geqslant 1,$$
$$\mathtt{k}_1 = 0, \quad \mathtt{k}_2 = 0, \quad \mathtt{l}_1 = 0, \quad \mathtt{l}_2 = 0,$$
$$\mathtt{e}_{12} = 0, \quad \mathtt{e}_{13} = 0, \quad \mathtt{e}_{23} = 0,$$

and the LPP has no solution. So these variables cannot be eliminated.

(3) As a last example let us choose f_{13}, f_{23}, l_2, the new constraints are

$$\mathtt{f}_{12} = 0, \quad \mathtt{f}_{13} \geqslant 1, \quad \mathtt{f}_{23} \geqslant 1,$$
$$\mathtt{k}_1 = 0, \quad \mathtt{k}_2 = 0, \quad \mathtt{l}_1 = 0, \quad \mathtt{l}_2 \geqslant 1,$$
$$\mathtt{e}_{12} = 0, \quad \mathtt{e}_{13} = 0, \quad \mathtt{e}_{23} = 0$$

and one solution is $\mathtt{f}_{13} = 2$, $\mathtt{f}_{23} = 2$ and $\mathtt{l}_2 = 1$, so the elimination of f_{13}, f_{23}, l_2 is possible.

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References

[1] José L. Bueso, J. Gómez Torrecillas, F.J. Lobillo, F.J. Castro, An introduction to effective calculus in quantum groups, in: Caenepeel, Stefaan et al. (Eds.), Rings, Hopf algebras, and Brauer groups. Proceedings of the Fourth Week on Algebra and Algebraic Geometry, SAGA-4, Antwerp and Brussels, Belgium, September 12–17, 1996, Marcel Dekker, New York, NY, Lect. Notes Pure Appl. Math. vol. 197, pp. 55–83, 1998.

[2] A. Kandri-Rody, V. Weispfenning, Non-commutative Gröbner bases in algebras of solvable type, J. Symb. Comput. 9 (1) (1990) 1–26.

[3] Viktor Levandovskyy, Noncommutative computer Algebra for polynomial algebras: Gröbner bases, applications and implementation, Ph.D. thesis, Universität Kaiserslautern, 2005.

[4] G.-M. Greuel, V. Levandovskyy, H. Schonemann, Plural. A Singular 3.0 Subsystem for Computations with Non-commutative Polynomial Algebras, Centre for Computer Algebra, University of Kaiserslautern, 2006. Available from: <http://www.singular.uni-kl.de>.

[5] Viktor Levandovskyy, PBW bases, non-degeneracy conditions and applications. Buchweitz, Ragnar-Olaf et al. (Eds.), Representations of algebras and related topics. Providence, RI: American Mathematical Society (AMS), Fields Institute Communications vol. 45, 2005, pp. 229–246.

[6] Viktor Levandovskyy, Intersection of ideals with non-commutative subalgebras, in: Proc. of the International Symposium on Symbolic and Algebraic Computation (ISSAC'06), ACM Press, 2006.

[7] José L. Bueso, José Gómez-Torrecillas, Alain Verschoren, Algorithmic methods in non-commutative algebra, Applications to quantum groups, In: Mathematical Modelling: Theory and Applications vol. 17, Kluwer Academic Publishers, Dordrecht, 2003 (English).

[8] George Lusztig, Quantum groups at roots of 1, Geom. Dedicata 35 (1–3) (1990) 89–114.

[9] C. De Concini, C. Procesi, Quantum groups, in: G. Zampieri et al. (Eds.), D-modules, Representation Theory, and Quantum Groups. Lectures Given at the Second Session of the Centro Internazionale Matematico Estivo (CIME) held in Venezia, Italy, June 12–20, 1992, Springer-Verlag, Berlin, Lect. Notes Math. vol. 1565, 1993, pp. 31–140.

[10] Corrado De Concini, Victor G. Kac, Representations of quantum groups at roots of 1, Operator algebras, unitary representations, enveloping algebras, and invariant theory, Proc. Colloq. in Honour of J. Dixmier, Paris/Fr. 1989, Prog. Math. vol. 92, 1990, pp. 471–506.

[11] J.L. Bueso, J. Gómez-Torrecillas, F.J. Lobillo, Re-filtering and exactness of the Gelfand–Kirillov dimension, Bull. Sci. Math. 125 (8) (2001) 689–715.

[12] George M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978) 178–218 (English).

[13] Lorenzo Robbiano, Term orderings on the polynomial ring, in: Bruno Buchberger (Ed.), Proceedings of EUROCAL'85, vol. 2, Lecture Notes in Computer Science, no. 204, Springer, 1985, pp. 513–517.

[14] G.-M. Greuel, G. Pfister, H. Schönemann, Singular 3.0, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2005. Available from: <http://www.singular.uni-kl.de>.
[15] V. Levandovskyy, F.J. Lobillo, O. Motsak, C. Rabelo, nctools.lib. A Singular 3.0 library with general tools for noncommutative algebras.
[16] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings. With the Cooperation of L. W. Small, Pure and Applied Mathematics, vol. XV, A Wiley-Interscience Publication, John Wiley & Sons, Chichester, 1987, 596pp.
[17] Bo Stenström, Rings of quotients, An introduction to methods of ring theory, Die Grundlehren der mathematischen Wissenschaften, Band 217, vol. VIII, Springer-Verlag, Berlin–Heidelberg–New York, 1975, 309pp.
[18] José L. Bueso, J. Gómez-Torrecillas, F.J. Lobillo, F.J. Castro, Primality test in iterated Ore extensions, Commun. Algebra 29(3) (2001) 1357–1371.
[19] J. Apel, W. Lassner, An extension of Buchberger's algorithm and calculations in enveloping fields of Lie algebras, J. Symb. Comput. 6 (2/3) (1988) 361–370
[20] J.L. Bueso, J. Gómez-Torrecillas, F.J. Lobillo, Homological computations in PBW modules, Alg. Rep. Theory 4 (2001) 201–218
[21] Hiroyuki Yamane, A Poincaré–Birkhoff–Witt theorem for quantized universal enveloping algebras of type A_N, Publ. Res. Inst. Math. Sci. 25 (3) (1989) 503–520
[22] H. Li, Noncommutative Gröbner Bases and Filtered-graded Transfer, Springer, 2002.