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# Constructions of bi-regular cages

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**Abstract**

Given three positive integers $r, m$ and $g$, one interesting question is the following: What is the minimum number of vertices that a graph with prescribed degree set $\{r, m\}$, $2 \leq r < m$, and girth $g$ can have? Such a graph is called a bi-regular cage or an $(\{r, m\}; g)$-cage, and its minimum order is denoted by $n(\{r, m\}; g)$. In this paper we provide new upper bounds on $n(\{r, m\}; g)$ for some related values of $r$ and $m$. Moreover, if $r - 1$ is a prime power, we construct the following bi-regular cages: $(\{r, k(r - 1)\}; g)$-cages for $g \in \{5, 7, 11\}$ and $k \geq 2$ even; and $(\{r, kr\}; 6)$-cages for $k \geq 2$ any integer. The latter cages are of order $n(\{r, kr\}; 6) = 2(kr^2 - kr + 1)$. Then this result supports the conjecture that $n(\{r, m\}; 6) = 2(rm - m + 1)$ for any $r < m$, posed by Yuansheng and Liang [Y. Yuansheng, W. Liang, The minimum number of vertices with girth 6 and degree set $D = \{r, m\}$, Discrete Math. 269 (2003) 249–258]. We finalize giving the exact value $n(\{3, 3k\}; 8)$, for $k \geq 2$.
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## 1. Introduction

Let $r \geq 2$, $g \geq 3$ be integers; an $(r; g)$-graph is an $r$-regular graph with girth $g$. An $(r; g)$-*cage* is an $(r; g)$-graph that has as few vertices as possible. The order of an $(r; g)$-cage is denoted by $n(r; g)$. For references on cages, see for instance the survey paper due to Wong [22], the survey paper due to Holton and Sheehan [16], or the website of Royle [21].

Let us denote by $D$ the degree set of a graph $G$. A $(D; g)$-graph is a graph having degree set $D$ and girth $g$. A $(D; g)$-*cage* is a $(D; g)$-graph that has as few vertices as possible. The number of vertices of a $(D; g)$-cage is denoted by $n(D; g)$. It is immediate that if $D = \{r\}$ then a $(D; g)$-cage is an $(r; g)$-cage. Erdős and Sachs [13] proved that $(r; g)$-cages exist for any regularity $r$ and any girth $g$. Using this result, Chartrand, Gould and Kapoor [11] pointed out the existence of $(D; g)$-cages. Downs, Gould, Mitchem and Saba [12] gave the following lower bound on $n(D; g)$

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where $D = \{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$:

$$n(D; g) \geq n_0(D; g) = \begin{cases} 1 + \sum_{i=1}^{t} a_k (a_1 - 1)^{i-1} & \text{if } g = 2t + 1, \\ 1 + \sum_{i=1}^{t-1} a_k (a_1 - 1)^{i-1} + (a_1 - 1)^{t-1} & \text{if } g = 2t. \end{cases} \tag{1}$$

When $D = \{r\}$, one can easily obtain $n(r; g) \geq n_0(r; g)$ by replacing $a_1$ and $a_k$ with $r$ in (1), where $n_0(r; g)$ is the known lower bound on the order of an $(r; g)$-cage (see [9] p. 105, or [10] p. 343). Some other structural properties of $(r; g)$-cages have been extended for $(D; g)$-cages, see [7]. A $(D; g)$-cage on $n_0(D; g)$ vertices is called a *minimal* $(D; g)$*-cage*. Thus, Kapoor, Polimeni and Wall [17] proved that a $(D; 3)$-cage is minimal because $n(D; 3) = n_0(D; 3) = 1 + a_k$.

When $D = \{r, m\}$, $2 \leq r < m$, then $(D; g)$-cages are called *bi-regular cages*. In the case $r = 2$, Chartrand, Gould and Kapoor [11] proved that the lower bound given by (1) is also attained by showing that $n(\{2, m\}; g) = n_0(\{2, m\}; g)$. They also proved that $(\{r, m\}; 4)$-cages are minimal because $n(\{r, m\}; 4) = n_0(\{r, m\}; 4) = r + m$ for any $r \geq 2$. In addition, it is known [12] that if $g \in \{5, 7, 9\}$, then $(\{3, m\}; g)$-cages are minimal, because $n(\{3, m\}; 5) = n_0(\{3, m\}; 5) = 1 + 3m$ and $n(\{3, m\}; 7) = n_0(\{3, m\}; 7) = 1 + 7m$ for all $m \geq 4$; and $n(\{3, m\}; 9) = n_0(\{3, m\}; 9) = 1 + 15m$ for $m \geq 9$.

Regarding girth $g = 6$, Yuansheng and Liang [23] showed the following lower bound:

$$n(\{r, m\}; 6) \geq 2(rm - m + 1), \tag{2}$$

for any $2 \leq r < m$. Also, in the same paper they conjectured that

$$n(\{r, m\}; 6) = 2(rm - m + 1) \quad \text{for all } r < m, \tag{3}$$

and proved the conjecture when $m - 1$ is a prime power, and also for $r = 3, 4, 5$ and any $m > r$.

In addition, upper bounds for the function $n(\{r, m\}; g)$ are provided by the authors in [2] for some related values of $r$, $m$ and even $g$. More specifically, the following results are obtained:

(i) If $3 \leq r < m$, where $m - 1$ is a prime power, and $g \in \{6, 8, 12\}$ then

$$n(\{r, m\}; g) \leq 2 + 2(r - 1)\frac{(m - 1)^{\frac{g}{2}-1} - 1}{m - 2}. \tag{4}$$

(ii) Let $r \geq 3$ and $k \geq 2$ be integers. Then for all $g \in \{4b + 2, 4b + 4\}$ with $b \geq 1$ we have
  (a) $n(\{r, k(r - 1) + 1\}; g) \leq kn(r; g) - 2(k - 1) \sum_{i=0}^{b} (r - 1)^i$;
  (b) $n(\{r, k(r - 1)\}; g) \leq kn(r; g) + 2(r - 1)^b - 2k \sum_{i=0}^{b} (r - 1)^i$.

When $g = 6$, by applying (ii) (a), using the lower bound (2) and taking into account that $n(r; 6) = n_0(r; 6) = 2(1 + (r - 1) + (r - 1)^2)$ for $r - 1$, a prime power (for this regularity the existence of a projective plane is known), the authors showed that $n(\{r, k(r - 1) + 1\}; 6) = 2k(r - 1)^2 + 2r$, for $r - 1$ a prime power and for all $k \geq 2$. This result supports Yuansheng's and Liang's Conjecture (3).

All the aforementioned exact values and the corresponding references are reviewed in Table 1.

In this paper we provide new constructions of bi-regular cages which allow us to obtain new upper bounds on $n(\{r, m\}; g)$ for related values of $r$, $m$ and $g$. First, we construct a minimal $(\{r, k(r - 1)\}; g)$-cage for $g \in \{5, 7, 11\}$, $r - 1$ a prime power and $k \geq 2$ an even integer. Then, we construct an $(\{r, kr\}; 6)$-cage for $r - 1$ a prime power and any integer $k \geq 2$, by contributing another example of a bi-regular cage that supports Yuansheng's and Liang's Conjecture (3). We finalize giving the exact value $n(\{3, 3k\}; 8)$, for $k \geq 2$, and showing a $(\{3, 6\}; 8)$-cage.

## 2. Results

Let $G = (V(G), E(G))$ be any graph, and let us denote the degree of any vertex $u \in V(G)$ by $\delta_G(u)$ and its neighborhood by $N_G(u) = N(u)$. We also use $\partial_G(u, v) = \partial(u, v)$ to denote the distance in $G$ between any two vertices $u$ and $v$. For any non-explicitly given terminology we refer the reader to [8,10].

Table 1
Exact values of $n(\{r, m\}; g)$ (Observe that $p^\alpha$ denotes *a prime power* and $(*)$ means *results obtained in this paper*)

| $n(\{r, m\}; g)$ | $g = 5$ | $g = 6$ | $g = 7$ | $g = 8$ | $g = 11$ |
|---|---|---|---|---|---|
| $r = 3$ | $3m + 1$<br>$m \geq 4$ [12] | $4m + 2$<br>$m \geq 4$ [23] | $7m + 1$<br>$m \geq 4$ [11] | $8m + \frac{m}{3} + 5$<br>$m = 3k(*)$ | |
| $r = 4$ | $4m + 1$<br>$m \geq 6$ even<br>$4m + 2$,<br>$m \geq 5$ odd<br>[18] | $6m + 2$<br>$m \geq 5$ [23] | | | |
| $5 \leq r < m$<br>$m - 1 = p^\alpha$ | | $2(rm - m + 1)$<br>[2,23] | | | |
| $3 \leq r < m$<br>$r - 1 = p^\alpha$ | $1 + rm$<br>$m = k(r - 1)$<br>$k \geq 2$even<br>$(*)$ | $2(rm - m + 1)$<br>$m = k(r - 1) + 1$<br>$k \geq 2$ [2], or<br>$m = kr, k \geq 2(*)$ | $1 + m(r^2 - r + 1)$<br>$m = k(r - 1)$<br>$k \geq 2$even<br>$(*)$ | | $1 + m\frac{(r-1)^3-1}{r-2}$<br>$m = k(r - 1)$<br>$k \geq 2$even<br>$(*)$ |

To compute the minimum number of vertices $n_0(r; g)$ of an $r$-regular graph of girth $g$, it is observed that the subgraph spanned by the vertices within distance $\lfloor (g - 1)/2 \rfloor$ from any vertex if the girth is odd, and from any edge if the girth is even, is a tree. In this way the following lower bound is obtained, which is the same as (1) replacing $a_1$ and $a_k$ with $r$:

$$n_0(r; g) = \begin{cases} 1 + r\dfrac{(r-1)^{\frac{g-1}{2}} - 1}{r - 2}, & \text{if } g \text{ odd;} \\ 2\dfrac{(r-1)^{\frac{g}{2}} - 1}{r - 2}, & \text{if } g \text{ even.} \end{cases} \tag{5}$$

If $g$ is odd the minimal $(r; g)$-cages are called *Moore graphs*, otherwise they are called *generalized polygons*. It is well known that Moore graphs exist only for $r = 2$ (cycles); $g = 3$ (complete graphs); or $g = 5$ and $r = 3, 7$ and possibly $r = 57$ (see [15]).

Generalized polygons exist if and only if $g \in \{4, 6, 8, 12\}$. If $g = 4$ then they are the complete bipartite graphs. The minimal $(r; 6)$-cages are known as *generalized triangles* and they are the incidence graphs of projective planes, which are known to exist if $r - 1$ is a prime power (although the existence question for other values of $r$ remains open). In the case $g = 8$, minimal cages are called *generalized quadrangles*, which are also known to exist when $r - 1$ is a prime power. Finally, when $g = 12$, *generalized hexagons* have also been constructed for $r - 1$, a prime power (see [8,14,19]). Other recent constructions for girth $g \in \{6, 8, 12\}$ can be found in [1,3–6].

Let $2 \leq r < m$ be integers and let $g$ be an odd integer. The lower bound (1) becomes

$$n_0(\{r, m\}; g) = \begin{cases} 1 + m\dfrac{(r-1)^{\frac{g-1}{2}} - 1}{r - 2} & \text{if } r \geq 3, \\ 1 + m\dfrac{g - 1}{2} & \text{if } r = 2 \end{cases} \tag{6}$$

which follows by replacing $a_k$ with $m$ and $a_1$ with $r$ in (1).

The next theorem gives an upper bound on the order of bi-regular cages of any girth in which the value of $m = m(r)$ depends on the value of $r$. This is done by taking an $r$-regular cage of girth $g + 1$ and order $n(r; g + 1)$.

**Theorem 1.** *Let $G$ be an $(r; g + 1)$-cage with $r \geq 3$ of order $n(r; g + 1)$. Then*

$$n(\{r, k(r - 1)\}; g) \leq \frac{k}{2}(n(r; g + 1) - 2) + 1$$

*for all even integer $k \geq 2$.*

**Proof.** Let $G$ be an $(r; g + 1)$-cage of order $n(r; g + 1)$, and let $C$ be a shortest cycle of length $g + 1$ in $G$ passing through an edge $xy$. Let $k \geq 2$ be an even integer, and let us consider $k/2$ disjoint copies of $G$, $\{G_i : i = 1, \ldots, k/2\}$,

$k/2$ corresponding shortest cycles $\{C_i : i = 1, \ldots, k/2\}$ of length $g+1$ each contained in $G_i$, and $k/2$ corresponding edges $\{x_i y_i : i = 1, \ldots, k/2\}$ belonging to each $C_i$. We construct a new graph $\Gamma$ by deleting from the union graph $(k/2)G$ all the vertices $\{x_i, y_i : i = 1, \ldots, k/2\}$, adding a new vertex $z$ and new edges joining $z$ to each vertex of $\cup_{i=1}^{k/2} \left(N_{G_i - y_i}(x_i) \cup N_{G_i - x_i}(y_i)\right)$.

We claim that $\Gamma$ is an $(\{r, k(r-1)\}; g)$-graph. Clearly, the degree of $z$ in $\Gamma$ is equal to $k(r-1)$ and the remaining vertices have degree equal to $r$. Moreover, the girth of $\Gamma$ is equal to $g$ because any cycle in $\Gamma$ either contains $z$, so its length is at least $g$ (namely, the path $C_i - \{x_i, y_i\}$ in $G_i - \{x_i, y_i\}$ is contained in a cycle of length exactly $g$ in $\Gamma$ passing through $z$), or it is entirely contained in a copy of $G_i - \{x_i, y_i\}$ which is a graph of girth at least $g+1$. Taking into account that the order of $\Gamma$ is $k/2\,(n(r; g+1) - 2) + 1$, the result follows. ■

As mentioned in the introduction, the exact value of the order of an $(\{r, m\}; g)$-cage for $g \in \{5, 7\}$ is only known for several particular cases, see Table 1. Now we are able to add more information concerning the exact value of $n(\{r, m\}; g)$ for $g \in \{5, 7, 11\}$. More precisely, in the following corollary of Theorem 1, the minimality of $(\{r, k(r-1)\}; g)$-cages is proved for $r-1$ a prime power, $k \geq 2$ an even integer and $g \in \{5, 7, 11\}$.

**Corollary 2.** *Let $r$, $k$ and $g$ be integers such that $k \geq 2$ is even, $r - 1 \geq 2$ is a prime power and $g \in \{5, 7, 11\}$. Then, $(\{r, k(r-1)\}; g)$-cages are minimal, that is,*

$$n(\{r, k(r-1)\}; g) = n_0(\{r, k(r-1)\}; g) = 1 + k(r-1)\frac{(r-1)^{\frac{g-1}{2}} - 1}{r-2}.$$

**Proof.** Let $g \in \{5, 7, 11\}$. From (6) it follows that

$$n(\{r, k(r-1)\}; g) \geq n_0(\{r, k(r-1)\}; g) = 1 + k(r-1)\frac{(r-1)^{\frac{g-1}{2}} - 1}{r-2}.$$

To see the other inequality, we recall that when $r-1$ is a prime power and $g+1 \in \{6, 8, 12\}$, minimal $(r; g+1)$-cages exist and by (5) have order

$$n_0(r; g+1) = 2\frac{(r-1)^{\frac{g+1}{2}} - 1}{r-2}.$$

Hence by applying Theorem 1, we have that

$$\begin{aligned} n(\{r, k(r-1)\}; g) &\leq 1 + k\frac{(r-1)^{\frac{g+1}{2}} - 1}{r-2} - k \\ &= 1 + k(r-1)\frac{(r-1)^{\frac{g-1}{2}} - 1}{r-2}. \end{aligned}$$

This completes the proof of Corollary 2. ■

The $(\{3, 4\}; 5)$-cage depicted in Fig. 1 has been obtained as described in the proof of Theorem 1; that is, considering one copy of the Heawood graph, i.e., the incidence graph of the Fano Plane. In this case $r = 3$, $k = 2$ and $g = 5$.

It is important to point out that these minimal bi-regular cages are not unique although it is well known that the minimal regular cages used in this proof are unique. For instance, in [11] the $(\{3, 4\}; 5)$-cage depicted in Fig. 2 was constructed. This cage has only one vertex of degree 4, like that depicted in Fig. 1. However, it is not isomorphic to the cage of Fig. 1. It is not difficult to prove this fact, since by deleting the vertex of degree 4 from the graph of Fig. 1 we obtain a bipartite graph (observe the vertex classes of black and white points). However, the graph obtained by deleting the vertex of degree 4 in the cage shown in [11] (see cage in Fig. 2) is not bipartite.

Furthermore, by applying Theorem 1 we obtain $(\{2, m\}; g)$-cages for an even integer $m$, starting from cycles of length $g+1$. The new graph $\Gamma$ is a $(\{2, m\}; g)$-cage formed by a set of $m/2$ cycles of length $g$ sharing only one vertex which has degree $m$. Then the remaining vertices have degree 2. These $(\{2, m\}; g)$-cages are not isomorphic to those constructed in [11], because they have exactly two vertices of degree $m$.

In the following theorem we also use $(r; g)$-cages to obtain upper bounds on the order of bi-regular cages for related values of $r$ and $m$.

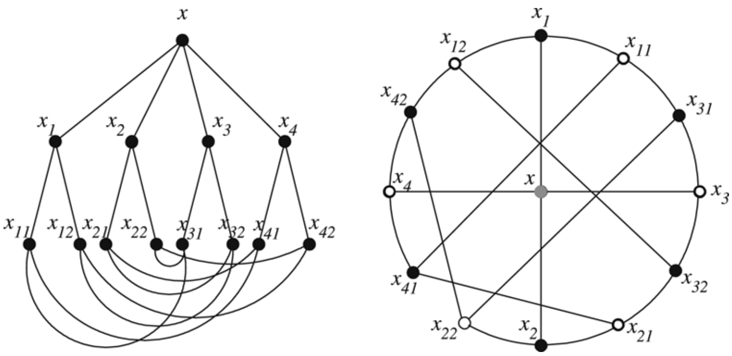


Fig. 1. Two drawings of a ({3, 4}; 5)-cage constructed by using the Heawood graph.

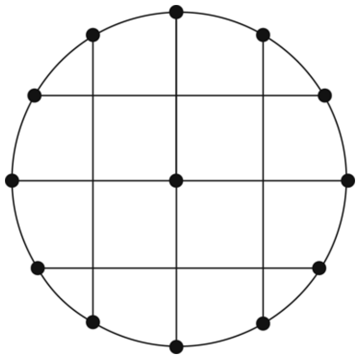

Fig. 2. The ({3, 4}; 5)-cage constructed in [11].

**Theorem 3.** *Let $r \geq 2$ and $k \geq 2$ be integers.*

(i) *If $g$ is even then $n(\{r, kr\}; g) \leq kn(r; g) - 2(k-1)$.*

(ii) *If $g$ is odd and there exists a minimal $(r; g)$-cage, then*

$$n(\{r, kr\}; g) \leq n_0(r; g) + (k-1)n(r; g+1) - 2(k-1).$$

(iii) *If $g$ is odd and $(r; g)$-cages are non-minimal, then*

$$n(\{r, kr\}; g) \leq kn(r; g) - 2(k-1).$$

**Proof.** (i) Let $G$ be an $(r; g)$-cage of even girth $g$. Let us consider $k$ disjoint copies of $G$, namely $\{G_i : i = 1, \ldots, k\}$, and $2k$ corresponding vertices $\{x_i, y_i \in G_i : \partial_{G_i}(x_i, y_i) = g/2, i = 1, \ldots, k, \}$. We construct a new graph $\Gamma$ by identifying all the vertices $\{x_i\}$ of the union graph $kG$ in a new one, denoted by $x$, and all the vertices $\{y_i\}$ in other new vertex, denoted by $y$ (see Fig. 3).

Clearly, $x$ and $y$ have degree equal to $kr$ in $\Gamma$, while the rest of the vertices have degree $r$. We claim that $\Gamma$ has girth $g$. In order to prove this, let us consider a cycle $C$ in $\Gamma$. Notice that we only need to study the case in which both $x$ and $y$ belong to $V(C)$, otherwise $C$ is contained in one copy $G_i$, so its length should be at least $g$. Hence suppose that $\{x, y\} \subseteq V(C)$ and $C = P_i \cup P_j$ where $P_i \subseteq G_i$ and $P_j \subseteq G_j$ are two $xy$-paths. However, as pointed out before, $\partial_{G_i}(x_i, y_i) = g/2$, thus the length of $C$ is at least $g$. Hence $\Gamma$ is an $(\{r, kr\}; g)$-cage of order $kn(r; g) - 2(k-1)$ and claim (i) follows.

(ii) Let $H$ be a minimal $(r; g)$-cage of odd girth $g$ and order $n_0(r; g)$. Let us consider $k-1$ copies of an $(r; g+1)$-cage $G$, namely $\{G_i : i = 1, \ldots, k-1\}$, and $2(k-1)$ corresponding vertices $\{x_i, y_i \in G_i : \partial_{G_i}(x_i, y_i) = (g+1)/2, i = 1, \ldots, k-1\}$. Likewise, we consider two vertices $x_H$ and $y_H$ on a shortest cycle in $H$ such that $\partial_H(x_H, y_H) = \frac{g-1}{2}$. Now, we construct a new graph $\Gamma$ by identifying in the union graph $(k-1)G \cup H$ all the vertices $x_i$ (resp. $y_i$) with the vertex $x_H$ (resp. $y_H$). We denote the new obtained vertices by $x$ and $y$ respectively. Observe that $x$ and $y$ have degree $kr$ in $\Gamma$, while the others are vertices of degree $r$. Reasoning as in point (i), we prove that $\Gamma$ has girth $g$. Thus $\Gamma$ is a $(\{r, kr\}; g)$-graph of order $n_0(r; g) + (k-1)n(r; g+1) - 2(k-1)$, so the result holds.

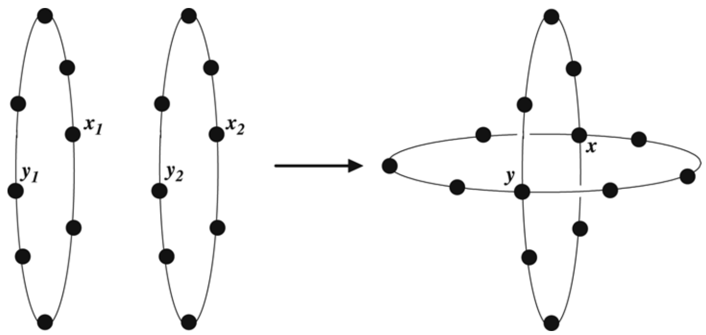


Fig. 3. A ({2, 4}; 8)-graph obtained from two copies of a (2; 8)-cage.

(iii) Let $G$ be a non-minimal $(r; g)$-cage of odd girth $g$. Hence the order $n(r; g)$ of $G$ satisfies $n(r; g) > n_0(r; g)$. Let $x$ be a vertex of $G$ and let us consider the tree $T$ spanned by the vertices within distance $(g-1)/2$ from $x$. The key of the proof is to note that $G$ must contain a vertex $y$ such that $\partial_G(x, y) = (g+1)/2$, otherwise $|V(G)| = |T| = n_0(r; g)$, which is impossible due to $G$ being a non-minimal cage. Let us consider $k$ disjoint copies of $G$, namely $\{G_i : i = 1, \ldots, k\}$, and $2k$ corresponding vertices $\{x_i, y_i \in G_i : \partial_{G_i}(x_i, y_i) = (g+1)/2, i = 1, \ldots, k,\}$. We construct a new graph $\Gamma$ as described in (i). It is immediate that the order of $\Gamma$ is $kn(r; g) - 2(k-1)$, the degree set of $\Gamma$ is $\{r, kr\}$ and there are exactly two vertices $\{x, y\}$ of degree $kr$. Let $C$ be a cycle in $\Gamma$. In the case $C$ is a cycle passing through $\{x, y\}$ then $C = P_i \cup P_j$ with $P_i \subseteq G_i$ and $P_j \subseteq G_j$ being two $xy$-paths. However, we know that $\partial_{G_i}(x, y) = (g+1)/2$ so the length of $C$ is at least $g+1$. In any other case, $C$ must be contained in one of the copies $G_i$ which is a graph of girth $g$. Hence $\Gamma$ is a $(\{r, kr\}; g)$-graph. This completes the proof. ■

By applying (2) and the existence of minimal $(r; 6)$-cages of order $n_0(r; 6) = 2(r^2 - r + 1)$ when $r - 1$ is a prime power, the following result is immediate by Theorem 3(i).

**Corollary 4.** *If $k \geq 2$ and $r - 1 \geq 2$ is a prime power, then*

$$n(\{r, kr\}; 6) = 2(kr^2 - kr + 1).$$

*Further, an $(\{r, kr\}; 6)$-cage is constructed by identifying in $k$ copies of an $(r; 6)$-cage one pair of corresponding vertices at distance 3, as in the proof of Theorem 3.*

Corollary 4 gives another example of $(\{r, m\}; 6)$-cages that supports the Conjecture (3) of Yuansheng and Liang (see [23]).

In what follows we want to improve Theorem 3(i) for the particular cases when $g \in \{8, 12\}$. With this aim, we introduce some basic definitions and results about generalized polygons (see [19,24]).

*A generalized $m$-gon of order $q$* is a point–line incidence geometry whose incidence graph is a $(q+1)$-regular bipartite graph with girth $2m$ and diameter $m$. Finite generalized $m$-gons exist only for $m \in \{3, 4, 6\}$.

As mentioned in the introduction, when $r - 1$ is a prime power, $(r; g)$-cages with $g \in \{8, 12\}$ are incidence graphs of 4-gons of order $r - 1$ and 6-gons of order $r - 1$, respectively. Following the geometric terminology, two elements in an $(r; g)$-cage $\Gamma$ are called *opposite* if they are at maximal distance from each other, i.e. $g/2$, $g \in \{8, 12\}$. Since $g/2 \in \{4, 6\}$ is even, clearly opposite elements are in the same bipartite set of $\Gamma$. An *ovoid $O$* in $\Gamma$ is a set of mutually opposite vertices (hence at distance $g/2 \in \{4, 6\}$) such that every element $v$ of $\Gamma$ belonging to the same bipartite set as the vertices in $O$, is at distance at most $g/4$ from at least one element of $O$. Under these hypotheses ovoids have cardinality $(r-1)^{g/4} + 1$ when $g = 8$, or $g = 12$ and $r - 1$ is an odd prime power different from 5 and 7, see [19,20], hence we have proved the following result.

**Proposition 5.** *Minimal $(r; g)$-cages with $r - 1$ a prime power and $g = 8$, or $r - 1$ an odd prime power different from 5 and 7 and $g = 12$, contain exactly $(r-1)^{\frac{g}{4}} + 1$ vertices which are mutually at distance at least $\frac{g}{2}$.*

**Corollary 6.** *If $k \geq 2$, $r-1$ a prime power and $g=8$, or $r-1$ an odd prime power, different from 5 and 7, and $g=12$, then*

$$n(\{r, kr\}; g) \leq 2k\frac{(r-1)^{\frac{g}{2}}-1}{r-2} - (k-1)((r-1)^{\frac{g}{4}}+1).$$

**Proof.** Let $G$ be a minimal $(r;g)$-cage of order $n_0(r;g)$ such that $r$ and $g$ have the properties of Proposition 5. Let us consider $k$ disjoint copies of $G$, namely $\{G_i : i = 1, \dots, k\}$ and $((r-1)^{g/4}+1)k$ corresponding vertices $\{x_i, y_i \in G_i : \partial_{G_i}(x_i, y_i) = g/2, i = 1, \dots, k\}$ as in the proof of Theorem 3(i). We construct a new graph $\Gamma$ following the same lines of reasoning described in the proof of Theorem 3(i). Hence $\Gamma$ is an $(\{r, kr\}; g)$-graph of order $k(n_0(r;g)) - (k-1)((r-1)^{g/4}+1)$ and the result follows. ■

In order to generalize the results of Yuansheng and Liang, first we make reference to the next result obtained in [2] for the case of even girth, which is an improvement of the lower bound (1), except when $g=6$ (in this case the best is the Yuansheng's and Liang's lower bound).

$$n(\{r, m\}; g) \geq \begin{cases} m+2+(mr-2)\dfrac{(r-1)^{\frac{g}{2}-2}-1}{r-2}+(r-2)(r-1)^{\frac{g}{2}-2} & \text{if } r \geq 4; \\ 1+\dfrac{(7m+3)2^{\frac{g}{2}-2}}{3}-m & \text{if } r = 3. \end{cases} \tag{7}$$

As a consequence of (7) and Corollary 6 we can present the following exact value for $g=8$.

**Corollary 7.** *If $k \geq 2$ then*

$$n(\{3, 3k\}; 8) = 25k + 5.$$

*Further, a $(\{3, 3k\}; 8)$-cage is constructed by identifying the five vertices of an ovoid in $k$ copies of a $(3; 8)$-cage.*

**Proof.** We obtain the upper bound by applying Corollary 6 for $r=3$, identifying 5 vertices of an ovoid in $k$ copies of a $(3, 8)$-cage of order 30. Thus we obtain a $(\{3, 3k\}; 8)$-graph with $25k+5$ vertices. The other inequality is obtained from (7), so the result holds. ■

## 3. Conclusions

In this paper we provide new constructions of bi-regular cages which allow us to obtain new upper bounds on $n(\{r, m\}; g)$ for related values of $r$, $m$ and $g$. We construct minimal $(\{r, k(r-1)\}; g)$-cages for $g \in \{5, 7, 11\}$, $r-1$ a prime power and $k \geq 2$ an even integer. We also construct an $(\{r, kr\}; 6)$-cage for $r-1$ a prime power and any integer $k \geq 2$, by contributing another example of a bi-regular cage that supports Conjecture (3) of Yuansheng and Liang. We conclude giving the exact value $n(\{3, 3k\}; 8)$, for $k \geq 2$, and showing a $(\{3, 3k\}; 8)$-cage. Taking into account this result, we pose the following conjecture.

**Conjecture 8.** *Let $m > 3$ be an integer. Then*

$$n(\{3, m\}; 8) = 8m + \left\lceil \frac{m}{3} \right\rceil + 5. \tag{8}$$

We believe also that the lower bound (7) can be improved for $r \geq 4$ and $g=8$. More precisely, we pose the following conjecture.

**Conjecture 9.** *Let $4 \leq r < m$ be integers. Then*

$$n(\{r, m\}; 8) \geq \frac{3m}{2} + (mr-2)r + (r-2)(r-1)^2. \tag{9}$$

If Conjecture 9 were true, then a similar reasoning as in the proof of Corollary 7 would provide the exact value of $n(\{4; 4k\}, 8) = 70k + 10$.

All the above results are summarized in Table 1.

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